ELEMENTARY HYDROSTATICS,

WITH

NUMEROUS EXAMPLES AND UNIVERSITY PAPERS.

BY

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PREFACE.

THE scope and object of this little book on Elementary Hydrostatics are explained in a few words. Although there are several text books on the subject in the English language, none seems fully to meet the wants, and to suit the capacities of Indian students qualifying for the B. A. degree. Those engaged in the work of teaching hydrostatics have often expressed dissatisfaction with the text books that are used in our colleges, and felt the necessity of better and more complete treatises. In the course of my duties as Professor of Mathematics in the Lahore College, I compiled, several years ago, the substance of the present work in the form of notes in order to assist my own pupils in their study of hydrostatics. I had then no intention of attempting to supply the want of a text book on the subject The publication of the present work is due to the encouragement I received from my friends Professor J. C. Oman, Professor of Science, and Professor T. C. Lewis M. A. Principal of the Lahore College, to whom the manuscript was shown. Indeed Professor Lewis has taken so much interest in the publication of the book as to see the greater part of it through the press, besides offering several valuable suggestions. The approbation of an eminent mathematician like Professor Lewis is perhaps the best assurance for the accuracy of the work and its fitness to fulfil the object for which it is prepared. To Professors Oman and Lewis, therefore, my best thanks are due.

In this work, I pretend to no originality, nor have I deviated much from the ordinary methods of treatment. My aim has simply been to present to the students in clear and concise language, and in forms best suited to their capacities, all the propositions of Elementary Hydrostatics which ought to be included in the course of studies prescribed for the B. A. degree examinations of Indian Universities. With this view, almost all the available English works on the subject have been consulted, and

the good features for which they are severally noted have been embodied in this work. It is accordingly hoped that the matter of this little book will be found to justify its publication, and to supply the want long felt among students.

As usual in text books, a number of illustrative exercises with answers have been given at the end of every chapter, in addition to several examples that have been worked out in the body of the book. Moreover, in order to furnish the student with a large variety of exercises, and also to give him an idea of the nature of the ordeal he has to go through in the examination hall, all the important problems set in the B. A. degree, L. C. E. and B. C. E. examinations of the Calcutta, Bombay and Madras Universities for a series of over twenty-three years have been appended. The volume concludes with a short history of the growth of the principles of hydrostatics, compiled chiefly from Whewell's History of the Inductive Sciences.

LAHORE
1st. January,
1888.

S. B. M.

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ELEMENTARY HYDROSTATICS.

CHAPTER I.

Definitions and First Principles.

1. The science of Hydrostatics treats of the conditions of the equilibrium of fluids and of the pressures they exert either on particles within their own mass, or on the surfaces of solid bodies in contact with them. It forms a part of the science of Statics, where a fluid appears as the chief means of transmission of force.

The laws of such transmission of pressure must, like the transmission of force by a rigid body, as a rod or a string, be established by experiment and observation.

2. Substances like water, oil, air or gas are called fluids. From ordinary experience, we learn that the particles of such substances are displaced by the slightest force, or that a portion of their mass, however small, can be very easily separated from the rest; and that they are characterised by their extreme divisibility.

By generalization from such experience, we obtain the idea of a perfect fluid, that is, a substance which possesses the above-named properties in the highest degree. Accordingly *fluid* is defined to be a collection of material particles in contact with each other, such that a mass of it can be very easily divided in any direction, and that a portion of it, however small or large, can be separated from the whole mass with the slightest force.

It follows from this definition that, if an indefinitely thin plate be made to divide a fluid in any direction, the resistance offered to the division in the direction of its plane is nothing, and that the pressure exerted by the fluid on the plane is entirely perpendicular to the plane.

It follows also that the pressure exerted at a point of a curved surface in contact with a fluid is in the direction of the normal to the surface at that point; for an indefinitely small area about the point may be regarded as a plane area coincident with the tangent plane at the point. This is called the "fundamental property of fluid."

3. The term *pressure* is used in hydrostatics in the same sense as it is in statics. It denotes a resisting force which the contiguous portions of the fluid exert upon each other, or upon rigid surfaces in contact with them.

4. That a fluid at rest is capable of exerting pressure, is verified by ordinary observation. A boat floating in water, the force necessary to immerse a piece of cork in water and the like, are all examples of the pressure exerted by the water on those bodies. The motion of boats under the action of currents, and that of windmills, are examples of pressure of fluid in motion.

5. A perfect fluid is an unrealizable conception, like a rigid or a smooth body in Mechanics ; as a matter of fact all fluids do more or less offer resistance to division or change of shape. All common fluids however when at rest very nearly fulfil the definition given above, and indicate the fundamental property of normal action. It is only in the case of fluids in motion, that the finite resistance, of the nature of friction, opposing change of shape at a finite rate, will cause marked disciepancy between the results of theory and observation.

6. Fluids are of two kinds, *liquids* and *gases*. The distinction between them consists in this, that the liquids, such as water, mercury &c. are almost incompiesible except unnder great pressures, and are comparatively inexpansible; while gases are easily compressible and expand freely and spontaneously. Hence liquids are distinguished from gases by their having what are called *free* surfaces. That liquids are really compressible has been proved by the experiments of Oersted, Colladon, Stuim and others The apparatus for this purpose employed by them is called *The Piezometer*.

The terms *elastic fluids*, and *inelastic fluids* are also frequently applied to gases and liquids respectively, but these terms are not quite accurate.

7. Liquids and gases are alike ponderable bodies, that is, they are acted on by gravity in the same way as solids. That gases such as air have weight, can be shewn directly by weighing a flask, first exhausted of air by means of an air-pump, and then by filling it with the gas and weighing it again.

8. When a plane is in contact with a fluid at rest, the pressure of the fluid upon the plane, if uniform at all points of the plane, is measured by the pressure exerted upon a unit of area of that plane.

For example, let a vertical cylinder with moveable base be filled with a liquid ; and suppose that a force of 20lbs is necessary to keep the base in its place at rest ; then 20lbs would be the

pressure on the base. If the pressure over the base be uniformly exerted, and the area of the base be 4 square inches, the pressure over every square inch would be 5 lbs. If the pressure on a square inch be taken as the unit of pressure, the pressure on the whole base is then said to be 4 units of pressure.

The phrase "*pressure at a point*" is conventionally used to express the pressure on a unit of area containing the point. A point being the indefinitely small portion of a surface, the pressure *on* a point will be evidently nothing. It is only when an infinite number of points coming in juxta-position form a definite area, that the pressure on the area becomes measurable. If the pressure be variable over the plane, as for instance on the vertical side of a vessel of water, the pressure *at any point* is measured by the pressure which *would be* exerted on a unit of area containing the point, supposing the pressure on the whole unit to be exerted at the same rate as in an indefinitely small area about the point. In other words, the expression "pressure at any point in a given direction" is used to denote the average pressure per unit of area containing the point, and perpendicular to the direction in question, when the area is indefinitely diminished. Thus if P be the pressure exerted on a small area a containing a given point, then $\frac{P}{a}$ will approximately express the *rate* of pressure over a; for a being very small, the pressure over it might be considered as sensibly uniform.

9. The pressure in a given direction at a point within the mass of a fluid will be best understood by imagining a rigid small plane area to be placed at right angles to the direction in question, and so as to contain the given point; and then conceiving that the fluid is removed from one side of the area. The force necessary to keep this plane at rest in that position is the measure of the fluid pressure at that point.

The introduction of this idea of rigid area for the measure of pressure does not introduce any new element of force; for we may imagine a portion of the fluid within a larger mass as separated from the rest by an indefinitely thin film, which may be supposed rigid, without affecting the relative position or equilibrium of the particles which form the exterior or interior portions of the fluid. It follows then that different portions of the fluid press against each other exactly in the way they would have done against rigid surfaces of the same form.

The idea of the action of the surrounding fluid at rest upon any portion within it *as if the latter were a rigid body*, is of importance in facilitating hydrostatical conception. This action may be taken as a fact deduced from experiment. But it also

follows from the fundamental property of fluid stated in Art. 2. For if we imagine this portion of fluid to become solid, the pressures of the surrounding fluid upon it, depending only upon the nature and position of the surface in contact with it, are not in any manner changed thereby. Nor is the pressure at any other point of the fluid changed by the imaginary solidification of that portion. This consideration enables us to apply the laws of statics to the equilibrium of fluids.

10. If no external force, such as gravity, act upon a fluid at rest, the pressure at every point in it will be the same.

For suppose the fluid to be contained in a closed vessel, the pressure at any point within depending only on the pressure exerted on the fluid by the sides of the vessel.

Take any two points A and B in the fluid and suppose that a portion of the fluid between them of the form of a cylinder with plane ends, to become solid. The actions of the surrounding fluid at every point of the curved surface of A and B being at right angles to the axis, it follows that the resultant pressures on the plane ends A and B, which are perpendicular to those ends, must be equal to one another. Similarly it is clear that the pressure at every point of the fluid is the same.

A homogeneous gas confined in a closed vessel serves to illustrate the practical bearing of the above case; for the action of gravity upon the gas is extremely small, and it follows that the pressures at different points within its mass differ from each other so minutely that no practical error will be committed in considering them all equal. The pressure in this case will depend upon the expansive power of the gas.

11. *The pressure at a point of a fluid at rest is the same in every direction.*

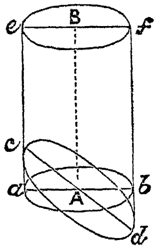
This statement asserts that if a very small plane area be placed in the fluid containing the point, the fluid pressure upon it will be independent of the direction of the area.

If the fluid be free from the action of external force, the truth of the statement follows directly from Art. 10. For the pressure in that case being constant throughout the fluid, the pressure on a given area will be the same at whatever inclination it is placed.

If the fluid be subject to the action of external forces, the statement is still correct, provided the area containing the point be taken very small.

The following formal proof may be given.

Let A and B be two adjacent points in the fluid. Describe a very thin cylinder about AB as axis, with plane ends ab and ef perpendicular to AB, the area of either being a. Suppose this cylinder to be solidified. The cylinder being very thin, the volume of the cylinder enclosed between ef and any oblique section cd through A, inclined at any angle θ to the section ab, will be same as the volume of the cylinder between the sections ab and ef.



The cylinder $cdef$ is kept at rest by the action of the external forces on its mass and the fluid pressures on its surface.

Let R be the resolved component along AB of the resultant external force on the cylinder, p and p' the pressures on units of area on cd and ef respectively, and let a' be the area of cd.

The cylinder being very thin, the pressures on cd and ef are practically uniform, and equal to $a'p$ and ap' respectively. The resolved part of $a'p$ along AB is $a'p \cos\theta$. The pressure of the fluid on the curved surface of the cylinder being all perpendicular to AB, we have for its equilibrium,

$$a'p \cos\theta = ap' \sim R.$$

But $a' \cos\theta = a$. $\therefore\ p = p' \sim \frac{R}{a}$, a result which is independent of θ.

Obs. When a fluid is subject to the action of external forces, the pressure within its mass varies from point to point, though the pressure at a given point is the same in every direction.

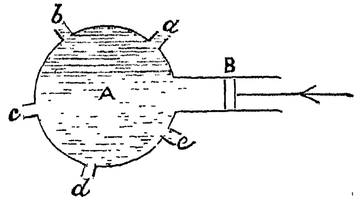
12. A pressure applied at any point of a homogeneous incompressible fluid at rest is transmitted equally to all parts of the fluid.

This proposition consists of two parts, (1) that the pressure applied is transmitted in *all* directions, (2) that it is *equally* transmitted; and may be established either directly by experiment or by deduction from the fundamental property of fluids enunciated in Art. 2.

The first part of the proposition may be illustrated thus:—

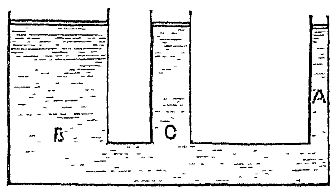
A cylinder provided with a piston B, is fitted into a hollow sphere A, in which small cylindrical jets a, b, c, d are placed perpendicular to the sides.

Let the sphere and the cylinder be filled with water and the piston B forced inwards. It will be found that the water spouts out through each of the jets, and not merely through the jet C, which is opposite to the piston.



Satisfactory illustration of the second part of the proposition can not be given; for the weight and the compressibility of the liquid and the friction of the piston can not be perfectly eliminated. An approximate illustration is given as follows.

A B and C are hollow cylinders of different diameters joined by a tube filled with water. Water-tight pistons are fitted in the cylinders, which are capable of moving up and down. Let A, B and C be the sectional areas of the pistons. If a weight P be placed on A, it is found that in order that the pistons at B and C may remain at rest, weights Q and R must be placed on them so that $\frac{P}{A}=\frac{Q}{B}=\frac{R}{C}$. Thus for example, if A be one square inch and B 20 square inches and a weight of 1 lb be placed on A, then 20 lbs must be placed on B in order to preserve equilibrium; and this is the case whether the cylinder C exists or not.



13. The equality of transmission of pressure may be deduced from the fundamental property of fluids.

Let A and B be two points in a fluid at rest. Describe a thin cylinder of fluid about AB as axis, and suppose this cylinder solidified. If p and p' be the pressures at A and B respectively and R the resolved component along AB of the resultant external force on the cylinder, then from the conditions of equilibrium as in Art. 11, we have,

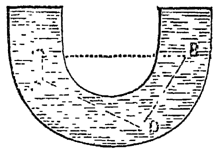
$$(p \sim p')a = R.$$

Now let the pressure at A be increased by q, and let q' be the increase of pressure at B in consequence; then since the force

acting upon the cylinder is not affected by this increased pressure at A or B, we have

$$p+q\sim(p'+q')=R$$
$$\therefore\ q=q$$

Obs. 1. If the line AB do not line entirely within the liquid, the points A and B may be connected by a series of lines all lying entirely in it, as AC, CD, DB. Then by repetition of the above reasoning we have the increment of pressure at A = that at C = that at D = that at B.



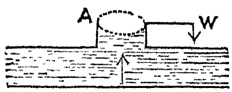
Obs. 2. By transmission of pressure to any point of the fluid is meant the increment of the normal pressure at the point.

Thus when a fluid is contained in a vessel, and the pressure at any point is increased, the additional pressure thereby transmitted to any point of the vessel or in the surface of an immersed body, must be along the normal at that point.

An oblique pressure at any point of the surface of the body may be resolved into two parts, one along the tangent and the other along the normal at that point; as the former has no action on the surface, the normal pressure needs only be considered.

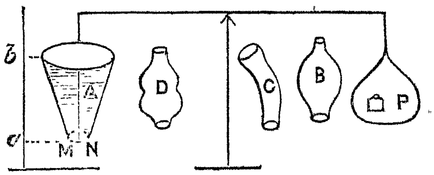
14. The property of equality of transmission of pressure through every part of the fluid is taken advantage of in the construction of various hydraulic machines, such as Bramah's press and others. But since their action depends upon a further principle of atmospheric pressure, a description of them is necessarily deferred till the principle of the Pump has been explained.

The construction of *safty valves* to preserve the boilers of steam engines from bursting under the pressure of the contained steam, also depends upon the principle of equal transmission of pressure. An appperture A is made in the side of a tube connected with the boiler, and is closed by a lid moveable about a hinge at one extremity A. A weight W is suspended from a rod attached to the lid and is so adjusted that it keeps the lid closed so long as a definite amount of pressure in the boiler (*i. e.* on a unit of area) calculated upon the strength of the boiler is not exceeded. When this pressure is exceeded, the excess of



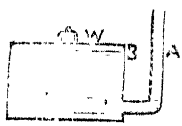
pressure is indicated by an equal increment at every point of the fluid, and the result is that the lid opens and a portion of the steam escapes, thereby diminishing the pressure in the boiler. When the pressure is sufficiently reduced the lid closes on account of the weight W.

15. An important conclusion follows from the principle of equal transmission of fluid pressure; namely that the pressure exerted by a liquid in virtue of its weight, on any portion of the liquid, or on the sides of the vessel containing it, does not depend upon the *form* of the vessel, nor upon the quantity of the liquid. An experimental illustration of this is given as follows :



A, B, C, D are vessels of different forms, all open at the bottom, the appertures at the bottom being in all cases equal, and such as may be exactly closed by the same disk MN. The vessel A is screwed on an upright stand ab, and the bottom is loosely closed by the disk MN which hangs on a thread at one end of a balance. When water is poured into the vessel to a height ab, let P be the weight which must be placed on the scale pan in order that the pressure of the water on the disk may be just counterbalanced. If the vessel A be replaced by B, C or D, and the water poured in exactly to the same height as before, it is found that the same weight P is required in each case to counterbalance the pressure on MN.

16. The toy known as the *hydrostatic bellows* is also another illustration of the same principle. B and C are the top and bottom planks of a bellows made of leather or any other flexible substance. CA is a tube leading into the bellows. When the bellows and the tube CA are filled with water, a very large weight upon B may be supported by a small pressure at A. In fact, if the sectional areas of A and B be A and B respectively, and P be the



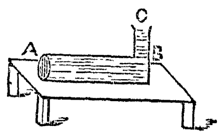
pressure applied on A, the pressure transmitted on B will be $\frac{B}{A}\cdot P$.

The pressure at A may be applied by various means; a simple closing of the apperture by the finger or by blowing in with the mouth, will serve the purpose. Or the weight of a column of water above A might be made to exert the pressure.

17. From a consideration of these principles, it readily follows, that a very small quantity of fluid is capable of producing a very considerable amount of pressure. For in the above figure, by making the tube CA very thin, *i. e*, making A very small in comparison to B, the quantity $\frac{B}{A} \cdot P$ may be made as large as we please, even when P is very small. By means of a very narrow column of water 40 feet high, Pascal succeeded in bursting a strong cask.

18. The fact that the pressure produced on the sides of a vessel does not depend upon the form of the vessel, and that by suitable arrangements this pressure may be made as large as we please by means of a very small quantity of fluid, gives rise to an apparent puzzle which is known as the hydrostatic paradox.

Let a closed vessel connected with a long thin vertical tube, be filled with water and placed upon a horizontal table. By making the tube $B\ C$ long and thin, the pressure on the bottom of the vessel may be made as large as we please.



This pressure is downward, and at first sight may be confounded with the pressure which the vessel itself exerts upon the table. For instance, if the area of the bottom be 100 square inches, and the sectional area of the tube be only $\frac{1}{10}$ of a square inch, an increment of pressure in BC by pouring in it a pound of liquid, would cause an aggregate increase of 1000 lbs pressure on the bottom of the vessel; and it would appear paradoxical that a pound of water will cause a pressure of 1000 lbs on the *table.* It must be remembered, that the pressure exerted on the bottom of the vessel is not all transmitted to the table. There will be a similar increase of pressure on the upper side of the vessel AB, and the difference of the downward pressure on the bottom and the upward pressure on the top of the vessel AB, would be the increase of pressure on the table. Consequently, the pressure on the table in any case is simply equal to the weight of the vessel and the water it contains.

19. The principle of equal transmission of pressure holds good also for gases; but the immediate effect of the additional

pressure will be to produce a compression of its volume. After the equilibrium is established the same additional pressure will have been transmitted to every portion of the fluid.

EXERCISES.

(1) What is the pressure of a fluid at a point? Show clearly how it can be proved experimentally that any pressure applied to the surface of a fluid is transmitted equally to all parts of the fluid.

(2) A square, whose side is $\frac{1}{3}d$ of an inch, is immersed in a fluid where the pressure at each point is 30 oz. per square inch. Find the pressure on the square. Ans. $3\frac{1}{3}$ oz.

(3) Two vertical cylinders fitted with pistons are connected at their lower ends by a tube. The area of the first piston being 12 square inches and the pressure applied to it 120lbs, what must be the area of the second piston, in order that the pressure transmitted to it may be one of 3 tons. Ans. 672 sq. in.

CHAPTER II.

Density and Specific Gravity.

20. It is usual in works on hydrostatics to give some description of the methods of measuring the mass and weight of fluids. There is however no difference between the measurement of the mass and weight of fluids and those of solids; what is said in treatises on mechanics about solid bodies applies *mutatis mutandis* to fluids.

21. We have seen that fluids are of two kinds, *viz.* gases and liquids. Again, howsoever fluids may differ from one another in their physical properties such as color, smell &c., they are all ponderable bodies, *i. e.* they are subject to the action of gravity; and as it is usual to estimate fluid pressures by the weight of a certain volume of the fluid itself, it may be convenient to indicate the method of estimating the weight of fluids.

22. The student of Dynamics must be familiar with the method of measuring force or weight either by means of the gravitation unit, or by means of the absolute or kinetic unit. The gravitation unit of weight is the weight (or force of gravity) in a given latitude of a given quantity of matter arbitrarily called the *unit of mass.* Thus the mass of a lump of platinum kept in

the Tower of London is arbitrarily taken as the British National unit of mass under the name of *a pound*. Its weight is also called a pound, and the weights of all other substances are measured by the ratio which they bear to this unit.

The absolu'e or kinetic unit of weight is a weight which is equivalent to the kinetic unit of force, *viz*. the force which acting upon the national unit of mass is capable of generating in a unit of time a unit of velocity.

Hence if g represent the accelerating effect of gravity in a given locality, the kinetic unit of weight in terms of the gravitation unit is $= \dfrac{\text{weight of the unit of mass}}{g}$.

23. The term *density* is used to express the quantity of matter in a body relatively to that of another. Thus if a cubic foot of a body A contains 5 times as much matter as is contained in a cubic foot of another body B, the density of A is said to be 5 times that of B.

The measure of density of a homogeneous body is defined to be the ratio which the mass of a given volume of the body bears to the mass of the same volume of another body to which it is referred, and which is hence called the *standard substance*.

The number one of course expresses the density of the standard substance itself.

The substance which is taken as the standard may be any whatever, but it is usual to take distilled water at its maximum density as the standard for all substances denser than water, and hydrogen for gases. The weight of a cubic foot of pure distilled water at a temperature of 62° Fahrenheit is about 998 oz, but for rough calculation, a cubic foot of water is assumed to weigh 1000 oz.

24. It is also usual to take the mass of a unit of volume of the standard substance as the unit of mass. Thus if one foot be the unit of length, the mass of one cubic foot of distilled water will be the unit of mass.

Let V be the volume of a substance whose mass is M, and whose density is ρ. Then it follows from the definition of density given above that a unit of volume of the substance contains ρ times the unit of mass; hence the volume V of the substance contains $V\rho$ times the unit of mass. Therefore $M = V\rho$.

Let W be the weight of the mass M expressed in the kinetic unit of weight; then

$$W = Mg \text{ or } g\rho V.$$

Obs. The kinetic unit of weight, *i. e.* the unit in which W is expressed, is the gth portion of the weight, in gravitation units, of a unit of volume of the standard substance The unit of volume depends upon the unit of length, and the value of g depends upon the units of length and time. The four units, *viz* those of length, time, weight and density, are therefore connected by one arbitrary relation, and accordingly if any three of them be arbitrarily fixed, the fourth is determined. For *e. g.* if a foot and and a second be units of length and time, and distilled water be the standard substance, then $g = 32 \cdot 2$ nearly.

$$\therefore \text{ unit of weight} = \frac{\text{weight of a cubic foot of distilled water}}{32 \cdot 2}$$

$$= \frac{1000}{32 \cdot 2} \text{ oz. nearly.}$$

25. When observation is confined to places in the same latitude and at the same distance from the center of the earth, the ratio of the masses of two bodies is the same as the ratio of their weights. Hence in such cases, it is more convenient to refer the weight of a homogeneous body of given volume to the weight of an equal volume of the standard substance. The term *Specific Gravity* is used to denote the relative weight of a substance as compared with that of the standard substance, and the measure of the specific gravity of a homogeneous substance is the ratio which the weight of a given volume of the substance bears to the same volume of the standard substance.

26. It would appear then that if the standard substance be the same, the number expressing the Density and the Specific Gravity of a given substance would be identical; for instance, if distilled water be the standard, the number 19·5 will express the density as well as the specific gravity of gold ; for

$$\frac{\text{mass of a cubic foot of gold}}{\text{mass of a cubic foot of water}} = \frac{\text{weight of a cubic foot of gold}}{\text{weight of a cubic foot of water}}$$
$$= 19 \cdot 5.$$

But if the standard substance be not the same, the density and the specific gravity of a substance will be expressed by different numbers.

27. Let V be the volume of a substance whose specific gravity as referred to a given standard is S; and let W represent its weight. Then by the definition of specific gravity given above, we have weight of a unit of volume of the substance $= S \times$ weight of a unit of volume of the standard substance ; hence if the body be homogeneous,

the weight of the volume V of the substance $= S.V. \times$ weight of a unit of volume of the standard substance.

It is usual in this case to take the weight of a unit of volume of the standard substance as the unit of weight; hence expressed in this unit,

$$W = SV.$$

Thus if distilled water be the standard, and one foot the unit of length,

unit of weight = weight of a cubic foot of water = 1000 oz.

$$\therefore \ W = SV.\ 1000 \text{ oz.}$$

28. It will perhaps be necessary to caution the beginner against an error he may be likely to commit with reference to the determination of weight by the equations $W = g\rho V$ and $W = SV$. He must particularly remember that the units of weight in these equations are different, and hence the weight W of the same substance will be represented by different numbers. For instance, if the standard substance be the same in both, and the unit of length the same, then the unit of weight

$$\text{in the first} = \frac{\text{weight of a unit of volume of standard substance}}{g},$$

and the unit of weight in the 2nd = weight of a unit of volume of standard substance. $\therefore \ \dfrac{\text{first unit of weight}}{\text{second unit of weight}} = \dfrac{1}{g}$.

The numerical value of W will of course vary inversely with the unit of weight employed; hence

$$\frac{\text{the number expressing } W \text{ by the first unit}}{\text{the number expressing } W \text{ by the second}} = \frac{g}{1}.$$

29. It may be useful here to indicate the relation between the English system of measurement and the *metrical* system of France. The beautiful harmony of the metrical system, and its decimal division and sub-division, have rendered it a favourite with the modern scientific men, and a knowledge of the system is indispensable for understanding scientific works of the present day.

In this system, the standard of length is the *metre*, which was originally defined as the ten-millionth part of the length of the quadrant of the earth's meridian, that is, the distance from the pole to the equator; but the metre is now practically

fixed by the distance between two marks on a standard rod. It is somewhat more than a yard, thus

One metre = 1·093633 yards = 39·370432 inches

One yard = ·914383 metre.

The metre is divided into ten equal parts called *decimetres* and a decimetre into ten equal parts called *centimetres*, and a centimetre into ten equal parts called *millimetres*. Thus

One decimetre = 3.9370432 inches = ·3280899 ft.

One centimetre = .3937043 in.

One millimetre = .0393704 in.

The unit of length having been thus settled, the unit of volume is the cube, an edge of which is the unit of length. Thus we speak of a *cubic metre* or *cubic decimetre*, and so on.

A cubic decimetre is also called a *litre* ; thus

One litre = ·035317 cubic foot

One cubic foot = 28·315312 litres.

In the metrical system, the mass of a litre of distilled water at 4°*C* is the unit of mass, called the *kilogram* ; and its weight is also called a *kilogram* (1000 grammes). A *gramme* is the mass of a centimetre of distilled water at 4°*C*. In India, the government standard *seer* is equivalent to a kilogram. The following relations have been found by experiment between the kilogram and the pound ;

One kilogram = 2·2036215 lbs

One lb = 453·5927 grammes

One gramme = 15·43235 grains.

It is superfluous to remark that in this system, the specific gravity of a substance will at once denote the weight of a litre of the substance in kilograms. For instance, when we say that the specific gravity of gold is 19·5, we at once know that the weight of a litre of gold is 19½ kilograms.

The unit of mass being thus settled, the kinetic unit of force is that force which is capable of producing a unit of acceleration in the unit of mass.

A force which acting on a mass of *one gramme* for one second generates a velocity of one centimetre per second, is technically called a *Dyne.*

Hence a *dyne* is the unit of force in what is called the *centimetre-gramme-second* or C. G. S. system of measurement.

It is found by observation that at Greenwich, the acceleration of the earth's gravity is 981·17 centimetres per second. Hence the force of gravity on a gramme at Greenwich is 981·17 dynes.

30. When a body is of such a nature that every unit of volume does not contain the same quantity of matter, in other words, when the density of different parts is different, the body is said to be Heterogeneous.

The formula $W=SV$ and $W=g\rho V$ will not be applicable to such cases.

If the density of a body varies continuously from point to point, the density *at any given point* expresses the density of a very small mass of the body about that point, it being assumed that the density throughout that small mass is practically uniform, and the same as at that point.

The mathematical conception of a continuously varying density is best formed in the following manner :—

Suppose a number of homogeneous strata of equal thickness, of densities $\rho, \rho_1, \rho_2, \rho_3 \cdots \rho'$ be arranged one upon another, so that the whole occupies a breadth AB. Suppose also that the quantities ρ_1 &c are in ascending or descending order of magnitude, each differing from the preceding by a very small quantity. If we now imagine the number of strata to become infinite, the whole breadth AB as well as the uppermost and lowermost strata remaining the same as before, and the breadth of each intermediate stratum indefinitely diminished, we shall have got a body satisfying the idea of a continuously varying density.

Any elastic heavy body like air, when at rest, will evidently assume the condition of a body such as that indicated above, the density increasing continuously from the top to the bottom. For the weight of the upper layers will partly compress the lower ones, thereby increasing the density at every successive point towards the bottom.

When the form of the body and the law of variation of its density are known, the weight of a given portion of it can be determined generally by the application of the Integral Calculus. We will here indicate only one of the simplest cases of the kind.

Example. A vertical cylinder of length h is filled with a fluid, whose density at any point varies directly as the depth of that point below the surface. To determine the weight of the fluid.

It is clear that the density at every point in the same horizontal plane is the same.

Divide the fluid into n horizontal layers of equal thickness, so that the breadth of each layer is equal to $\frac{h}{n}$.

The depth of the r^{th} layer $=\frac{hr}{n}$, and its density may therefore be denoted by $\mu\frac{hr}{n}$, and its weight $=g\mu\frac{hr}{n}\cdot A.\frac{h}{n}$, A being the sectional area of the cylinder, and μ the constant of variation,

$$\therefore \text{ weight of the } n \text{ layers of fluid } = Ag\mu\frac{h^2}{n^2}\ (1+2+3+\ldots\ldots+n)$$

$$=Ag\mu\frac{h^2}{n^2}\cdot\frac{n(n+1)}{2}=\tfrac{1}{2}Ag\mu h^2\left(1+\frac{1}{n}\right).$$

Now let n be made infinite, so that the breadth of each layer becomes indefinitely thin, and the liquid approaches to the idea of a continuously increasing density varying with the depth.

$$\therefore \text{ weight of the fluid} = \tfrac{1}{2}Ag\mu h^2.$$

If ρ denote the density of the fluid at the bottom of the cylinder,

$$\rho=\mu h$$

$$\therefore \text{ weight of the fluid} = \tfrac{1}{2}Ag\rho h.$$

The case of a fluid with density varying as the square or cube of the depth may be similarly treated.

31. The subject of units presents some difficulty to the beginner by reason of its simplicity. We will accordingly close the chapter by illustrating the subject in some special questions. The student should remember that if g represent the numerical value of gravity when s ft. and t seconds are units, then

$$g=\frac{t^2}{s}\times 32{\cdot}2 \text{ nearly, or } =\frac{t^2}{s}\times 32 \text{ roughly.} \quad (\textit{vide Dynamics}).$$

Example 1. In the equation $W=g\rho V$, if water be the standard substance, determine the unit of length that the weight may be given in pounds, the unit of time being one second.

As explained in Art. 24, we have from the equation $W=g\rho V$, weight of a unit of volume of the standard substance

$$=g\times\text{unit of weight.}$$

$\therefore$ if x ft. be the required unit of length, we have

weight of x^3 cubic feet of water $=g$ lbs.

But with this unit of length, $g=\dfrac{32}{x}$.

$$\therefore\ x^3.\frac{1000}{16}=\frac{32}{x},\ \text{or } x^4=\frac{32\times 16}{1000}.$$

From this x becomes known.

Example 2. If the units of weight, length and time be **1 lb**, 1 yard, and $\frac{1}{2}$ a second respectively, compare the standard substances in the formulæ $W=g\rho V$ and $W=SV$.

$$\text{Here } g=\frac{32\times\frac{1}{4}}{3}=\frac{8}{3}.$$

In $W=g\rho V$, the weight of a cubic yard of the standard substance $=\frac{8}{3}$ lbs.

And in $W=SV$, the weight of a cubic yard of the standard substance referred to $=1$ lb.

$$\therefore\ \frac{\text{specific gravity of the 1st standard}}{\text{specific gravity of the 2nd standard}}$$

$$=\frac{\text{weight of a cubic yard of 1st standard}}{\text{weight of a cubic yard of 2nd standard}}=\frac{8}{3}.$$

EXERCISES.

N. B. Unless otherwise stated, water is to be taken as the standard substance, and the acceleration due to gravity $=32$, with one foot and one second as units.

(1) Compare the units of weight in the formulæ $W=g\rho V$ and $W=SV$, when 2 feet and 2 seconds are the units of length and time. Ans. 1 : 64.

(2) Find the weight in pounds of a cubic foot of silver, the specific gravity of which is 10·5. Ans. 656·25.

(3) If one inch be the unit of length, what must the unit of *time* be, that the formula $W=g\rho V$ may give the weight in ounces. Ans. $\frac{5}{12}\sqrt{(15)}$.

(4) If one inch and one hour be the units of length and time, find the ratio of the units of weight in the two formulæ $W=SV$ and $W=g\rho V$. Ans. 138240 : 1.

(5) Taking water as the standard substance, and a yard as the unit of length, what is the unit of weight in $W=SV$? and what is the weight of a cubic foot of a substance whose specific gravity is 1·6? Ans. 27000 oz.; 100 lbs.

(6) Find the volume, in cubic inches, of a piece of iron weighing 14 lbs (S. G. of iron = 7·8). If a cubic foot of the standard substance weighs 4 lbs, what must be the unit of length in order that the equation $W=SV$ may give the weight in ounces? Ans. 49·624; 3 inches.

(7) The specific gravity of a mixture of equal volumes of fluids B and C is 1·2, of C and A is 1·1, and of fluids A and B is ·8; find the specific gravities of A, B and C respectively. Ans. ·7, ·9, 1·5.

(8) A coin composed of platinum and silver is found to be exactly of the same size and weight as a sovereign; find the relative weight of the two metals in it, the specific gravities of platinum, silver and gold being 21, 10·5, and 17·5 respectively. Ans. 4 : 1.

(9) A certain volume of the standard fluid is mixed with 2 cubic inches of a fluid whose specific gravity is 1·5; the specific gravity of the mixture is 1·2; find the volume of the standard fluid. Ans. 3 cub. in.

(10) The specific gravity of pure gold is 19·3, and of copper 8·62; required the specific gravity of *standard* gold, which is a mixture of eleven equal parts by weight of gold and one of copper. Ans. 17·6 nearly.

CHAPTER III.

Equilibrium of Fluids.

32. To find the condition of equilibrium of a fluid acted on by external forces.

If no external force act upon the fluid, the pressure is the same at every point. (*vide Art 10*). But when such forces act upon it, the pressure varies from point to point.

The expression *impressed force* is used to denote a force other than resisting forces due to the mutual actions of the

particles of the body acted upon, or due to the actions of the surrounding medium upon the body.

Let A be a point in the fluid, and B another point adjacent to it. Describe a thin cylinder of fluid bounded by planes perpendicular to the axis AB, and consider the cylinder solidified.

Let p and p' be the average pressures on a unit of area on the ends A and B; a = area of A or B, l = length of the cylinder, and ρ = the density of the fluid.

The impressed forces together with the resisting forces keep the body in equilibrium. They may all be resolved along AB and two lines at right angles to AB. The pressures of the surrounding fluid on the curved surface of the cylinder, being everywhere normal to the surface, have no resolved parts along AB. Hence if R denote the sum of the resolved parts, along AB, of the impressed forces acting upon the cylinder, we must have, for equilibrium in the direction of AB,

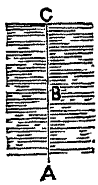
$$R = (p \sim p')a.$$

If the impressed forces depend upon the mass of the fluid acted upon, (*e. g.* the earth's gravitation, magnetic attraction &c.) let Q represent the component force along AB on a unit of mass, then $R = Qal\rho$.

$$\therefore \quad Q\rho = \frac{p \sim p'}{l}.$$

This shews that if equilibrium be possible under the given system of forces, the rate of increase of pressure (*i. e* per unit of length) in any given direction, is equal to the intensity of the resolved component, in that direction, of the resultant impressed force per unit of volume of the fluid.

33. To find the pressure at a point in a homogeneous heavy fluid at rest.



Take the points A and B in a vertical line at a distance l apart. Then observing that here gravity is the only impressed force, that it acts vertically downwards, and its magnitude is $g\rho.a.l$, where ρ is the constant density of the liquid, we have by the preceding article,

$$(p - p')a = g\rho al$$

$$\text{or } p - p' = g\rho l$$

Hence *the difference of the pressures at any two points in the same vertical line in a homogeneous liquid at rest varies as the vertical distance between them.*

Again, let AB cut the *free surface* of the liquid at C.

Let $AC = z$. At C the pressure being zero,

$$\therefore\ p = g\rho z.$$

i. e. The pressure at any point in a homogeneous liquid at rest varies as the depth of the point below the free surface of the liquid.

Obs. 1. The expression $g.\rho.z$ denotes the weight (in kinetic units) of a cylinder of the liquid whose length is z, and whose transverse section is a unit of area.

Now, $g \times$ kinetic unit of weight

$=$ weight (in gravitation units) of a unit of volume of the standard substance, (*vide Art* 24)

$= U$ suppose.

Hence expressed in gravitation units,

$$p = \rho z.\ U.$$

Also if w represent the weight of a unit of volume of the given liquid, $w = \rho U$. Hence $p = wz$.

Example. Find the pressure at the depth of 20 inches of a fluid whose density referred to water is ·8.

To express p in *kinetic units*, the units of length and time must be given. Let these be one inch and one second respectively. Then

$$g = 32{\cdot}2 \times 12 = 386{\cdot}4.$$

$\therefore$ Kinetic unit of weight, which is here equivalent to a force producing acceleration of velocity of an inch per second in a cubic inch of water, is equal to $386{\cdot}4th$ part of the weight of a cubic inch of water, *i. e.* $\dfrac{1000}{386{\cdot}4 \times 1728}$ oz.

$\therefore$ The required pressure on a square inch

$= 386{\cdot}4 \times {\cdot}8 \times 20$ kinetic units

$= 6182{\cdot}4$ kinetic units.

Or the pressure $= {\cdot}8 \times 20 \times \dfrac{1000}{1728}$ oz.

$= 9{\cdot}26$ oz. nearly.

Obs. 2. The principle of the reasoning in this article will be as well applicable to liquids of varying densities, and also to elastic fluids, as gas or air; thus

the pressure at A = weight of the cylinder AC of the liquid + pressure at C.

But the weight of the cylinder AC can not be in this case represented by $g\rho z$, but must be treated as in Art. 30.

Obs. 3. The so called *free surface* of a liquid is ordinarily subject to the atmospheric pressure, which is known from observation to be about $14\frac{1}{2}$ lbs. on a square inch at the level of the sea. Hence if π denote the atmospheric pressure on a unit of area,

$$\text{the pressure at } A = \pi + g\rho z.$$

34. *The pressure at any point of a horizontal plane in a heavy liquid at rest is the same.*

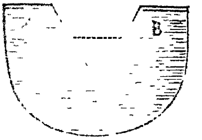
Let A and B be any two points in a horizontal line in the liquid ; p and p' the pressures at those points.

∴ Since gravity, which is the only impressed force on the liquid, has no resolved part along AB, we have by Art. 32,

$$p - p' = 0, \text{ or } p = p'.$$

Obs. 1. The points A and B must be taken to lie in the *same* liquid, although the line AB need not lie entirely in it. Moreover the pressures at A and B shall be equal for all inclinations of the small areas containing these points. *See Art.* 11.

Obs. 2. If the form of the vessel which contains the liquid be such that the line AB does not lie entirely in the liquid, we may join A and B by a series of vertical and horizontal lines which lie entirely in the liquid, as AC, CD, DB.



Now, pressure at C = pressure at D, and $AC = BD$.

By Art. 33, pressure $C = p + g\rho . AC$
and pressure at $D = p' + g\rho . BD$

$$\therefore p = p'.$$

35. *The free surface of a liquid at rest is a horizontal plane.*

This follows directly from the preceding article ; for on the free surface, either there is no pressure, or every point is exposed to the same atmospheric pressure ; hence the free surface cannot but be a horizontal plane.

This is more formally proved thus :

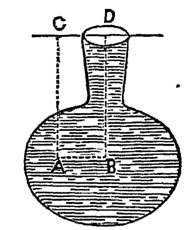
Take any two points A and B in the same horizontal plane in the liquid, and let the vertical lines AC, BD cut the free surface at C and D. Then since the pressure at A = pressure at B,

$$g\rho AC = g\rho . BD, \text{ or } AC = BD.$$

Similarly it may be proved that the vertical depth of the horizontal plane below the free surface is everywhere the same.

Hence the free surface is a horizontal plane.

36. When a liquid is contained in a vessel of any form, the free surface is the horizontal plane passing through the highest point of the liquid.



If the form of the vessel be such that the vertical line AC through a point A in the liquid does not lie entirely in the liquid, the pressure at A will still be $g\rho AC$; for the point A may be connected with the free surface by a series of vertical and horizontal lines, and the sum of the vertical depths equals AC. In other words, the pressure at A is that due to a depth AC, exactly as if AC were entirely in the liquid.

37. The pressure of the liquid at a given depth may be illustrated by the following experiment. Take a hollow cylinder open at both ends. A heavy flat disc is held by means of a string fastened to the disc against the bottom of the cylinder, thereby exactly closing that end. Holding the string tight, the cylinder is immersed in water. It is found that when a certain depth is attained, the string may be loosened and the disc will be supported by the upward pressure of the water upon it.

Let x be the depth of the disc when just supported, A its area, and W its weight ;

then, W = upward pressure of water = weight of a volume Ax of water, from which x is determined.

If the disc be lowered further down, and then water be very gently poured into the cylinder, the disc will sink down only when the surface of the water inside the cylinder is at the position of the disc when it was just supported. This shews that the upward pressure on the disc increases with the depth of the disc.

Obs. The atmospheric pressure on the disc will not affect its position of equilibrium; for this pressure exerted on the exposed surface of the water is equally transmitted to all points of the liquid, and the two faces of the disc are equally affected by it.

38. Two homogeneous liquids that do not mix rest one upon another. To determine the pressure at a given point A in the lower liquid.

Let the vertical line through A cut the common surface of the liquid at B, and the free surface at C.

Let ρ and ρ' be the densities of the upper and the lower liquids respectively.

Then by reasoning as in Art. 33, we have

$$\text{pressure at } A = \text{weight of the column } AC$$
$$= g\rho\, BC + g\rho' \cdot AB.$$

If the surface at C be exposed to atmospheric pressure, this pressure will be transmitted to every point of the liquids. Hence if π be its measure on a unit of area,

$$\text{Pressure at } A = \pi + g\rho BC + g\rho' AB.$$

The same reasoning may be extended to any number of liquids lying one upon another.

39. *The common surface of two liquids that do not mix, is a horizontal plane.*

Take any two points E and F in the same horizontal plane in the lower liquid, and let the vertical lines through them cut the common surface at C and D, and the free horizontal surface of the upper liquid, at A and B.

$$\text{Now pressure at } E = g\rho AC + g\rho'.CE$$
$$\text{and pressure at } F = g\rho.BD + g\rho' DF.$$

$$\text{But the pressure at } E = \text{pressure at } F$$

$$\therefore\ \rho.AC + \rho'.CE = \rho.BD + \rho' DF$$
$$\text{also } AC + CE = BD + DF$$

$\therefore$ multiplying by ρ' and subtracting, we have

$$(\rho' - \rho) AC = (\rho' - \rho) BD, \text{ or } AC = BD.$$

In the same manner it may be shown that the depth of the common surface below the free horizontal surface AB, is the same at every point; therefore the common surface is horizontal.

Obs. In consequence of the extreme mobility of the particles of the fluids, the heavier fluid will rest on the bottom of the vessel, and the lighter one will lie upon it. If we find a drop of water floating in a cup of oil, it is because neither water nor oil satisfies our idea of a perfect fluid, to which alone the theoretical reasonings of hydrostatics are applicable; the weight of a particle of water being insufficient to overcome the adhesion between two contiguous particles of oil.

40. The locus of points, in a fluid, at which the pressures are the same, is called *the surface of equal pressures*. To investigate the general nature of the surface of equal pressure when it exists, is beyond the limits of elementary hydrostatics. One particular property of it however may be indicated as following immediately from the equation $(p \sim p')\,a = R$, of Art 32.

If A and B be the two contiguous points on a surface of equal pressure, we have $p = p'$, $\therefore R = 0$, which shews that the resultant impressed force on any particle of fluid situated on a surface of equal pressure has no resolved component *in any direction whatever along the surface*.

Therefore the resultant impressed force, if any, must be normal to the surface at that point. Hence a surface of equal pressure is such that it cuts at right angles the lines of action of the resultant impressed forces on the several particles of fluid constituting that surface.

Thus, if gravity be the only impressed force, it follows at once, that the surfaces of equal pressure must be spherical surfaces concentric with the earth; for they are everywhere perpendicular to the directions of gravity, *i. e.* to the radii of the earth. This is approximately the case with the sea at rest, the difference of temperature at different places being neglected; for the atmospheric pressures on its surface being transmitted equally in all directions do not affect the reasoning of this article. When we confine our attention to a small quantity of such liquid, the corresponding small portions of the surfaces of equal pressure may be regarded as horizontal planes. Compare with Art 34.

Hence also the free surface of a heavy liquid at rest which is necessarily a surface of equal pressure, (the pressure at every point being either zero or constant atmospheric pressure) must be also spherical, and a small portion of it may be regarded as a horizontal plane. (*See Art.* 35).

41. *In a fluid at rest under the action of gravity alone, the surfaces of equal pressure are also surfaces of equal density.*

For consider two surfaces of equal pressure infinitely near to each other. Let the fluid between them be divided into very thin cylinders of equal transverse sections, the axes being perpendicular to the surfaces. The volumes of the cylinders are equal.

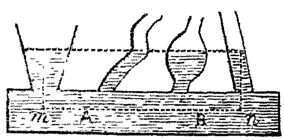
The difference of the pressures on the ends being the same for each cylinder, the impressed force of gravity on each must also be the same (Art. 32).

But as the magnitude of the impressed force varies as the mass of the fluid acted upon, the mass and therefore the density of every particle of fluid is the same throughout the layer under consideration.

The same result is true for all natural forces, although a strict demonstration of the subject is beyond the range of this elementary treatise.

42. When several vessels of any form whatever, containing the same liquid (as water), communicate with each other, it is necessary for equilibrium of the whole, that the liquid be in equilibrium in each vessel, and that the free surfaces of the liquid in all the vessels be in the same horizontal plane.

For let there be a number of vessels of various shapes, all connected together by the vessel *AB*.



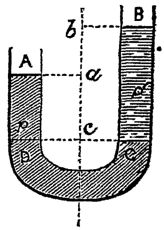
Pour in liquid, which after filling *AB*, rises in each of the vessels.

Consider any horizontal plane *mn* through the liquid. The pressure at every point on the plane *m n* is the same; and as the pressure at any point is proportional to the vertical depth of that point below the free surface of the liquid, it follows that the free surface of the liquid in each vessel must be at a constant vertical height above *m n*. In other words, they are in the same horizontal plane. This principle is sometimes stated in the following terms, "Liquids maintain their level".

An important practical illustration of this principle is seen in the supply of water in towns or workshops from a common reservoir placed at a height, from which water may be supplied by means of pipes to vessels at various distances connected

with the reservoir, provided that the vessels and the pipes be not anywhere higher than the surface of the water in the reservoir.

43. If two homogeneous liquids of different densities that do not mix, be contained in separate vessels communicating with each other, they will be in equilibrium when the vertical heights of their free surfaces above their horizontal surface of contact are in the *inverse ratio* of their densities.



Let AC and BC be the two liquids of densities ρ and ρ' respectively, C being the common surface of contact.

Since the liquids are in equilibrium, the downward pressure of the liquid BC at the point C, must be counteracted by the upward pressure of the liquid AC at the same point.

Let the horizontal plane through C cut the liquid at D.

The pressure at C due to liquid ρ = pressure at D

$$= g\rho . ca.$$

And the pressure at C due to the liquid $\rho' = g\rho' bc$,

$$\therefore \; g\rho . ac = g\rho' bc$$

$$\therefore \; \frac{ac}{bc} = \frac{\rho'}{\rho},$$ which proves the proposition.

It must be noticed that in cases like this, equilibrium can not exist unless the heavier liquid rest partly in both the vessels.

Also the existence of atmospheric pressure does not affect this relative height; for that pressure transmitted equally through the liquid from the surface of each liquid does not affect the equilibrium of the particles at C, the difference of the atmospheric pressure for the difference of the levels of A and B being of course neglected.

EXERCISES.

(1) In two uniform fluids, the pressures are the same at depths 3 and 4 inches respectively. Compare the pressures at the depths 7 and 8 inches respectively. Ans. 7 : 6.

(2) To what extent is the pressure on the base of a vessel affected by pouring in more liquid ?

(3) A house is supplied with water from a tank which is 60 feet above the level of the ground floor; find the pressure of the water at a height of 25 feet above the ground floor.

Ans. 35,000 oz. on a sq. ft.

(4) Prove that the surface of a liquid at rest is a horizontal plane. Where would your proof fail if the surface of the liquid were large?

(5) Two liquids whose specific gravities are as 2 : 1, rest in a cylindrical tube of radius r, whose vertical section is three sides of two concentric squares, of which the sides are a and $a+2r$ respectively. One of the liquids has its surface at the top of one side of the tube, and the common surface is half way down the side a; determine the volume of the other liquid. The parallel portions of the tube are vertical. Ans. $\frac{1}{4}\pi r^2(9a+8r)$.

(6) A reservoir of water is 162 feet above the ground floor of a house; a pipe running from it is closed in a room 18 feet above the ground floor, by a horizontal metal plate equal in area to one square inch; show that the pressure on the plate may be balanced by a force equal to the weight of a cubic foot of water.

(7) A tube of small uniform bore is in the form of an equilateral triangle whose plane is vertical and base horizontal. It is exactly filled with equal volumes of 3 fluids which do not mix, whose densities are as 1, 2 and 3. Shew that in the position of equilibrium, each fluid occupies $\frac{2}{3}$ of one side and $\frac{1}{3}$ of the other.

(8) If 30 feet be the height of the water-barometer, find the depth below the surface of a tank of a point where the pressure is double the pressure at a depth of 10 feet. Ans. 50 ft.

(9) A hollow cone, whose axis is vertical and base downwards, is filled with equal volumes of two liquids of densities 2ρ, and ρ. Prove that the pressure at a point in the base is $(2-2^{-\frac{1}{3}})$ times as great as when the vessel is filled with the lighter fluid.

(10) If 2ρ and 3ρ be the densities of two fluids, and the lengths of the arms of a bent tube in which they meet be a and b respectively, prove that in order that the tube may be completely filled, the height of the column of the lighter fluid above the horizontal plane in which they meet, must be $3(a-b)$.

(11) If a parallelogram be immersed in a fluid in any manner, the sum of the fluid pressures at the extremities of one diagonal is equal to the sum of those at the extremities of the other.

(12) If the pressure at the depth of 32 feet below the surface of water be 15 lbs to the square inch, what will be the pressure on a square foot at the depth of 10 feet 8 inches.

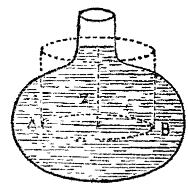
Ans. $826\frac{2}{3}$ lbs.

CHAPTER IV.

Total Pressures on Immersed Surfaces.

44. A horizontal plane lamina of area A is immersed in a heavy liquid at rest. To find the pressure upon it.

Let z be the depth of any point of the lamina below the horizontal free surface of the liquid; which it must be remembered, passes through the highest point of the liquid.



Let ρ be the density of the liquid, and p the pressure on a unit of area in A, which is constant throughout A, then

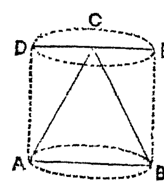
$$p = g\rho . z.$$

$\therefore$ the pressure on $A = A . p = g\rho . A . z.$

Now $A . z.$ denotes the volume of a cylinder whose sectional area is A, and length of axis equal to z.

Hence the pressure on A may be expressed as equivalent to the weight of a cylinder $A.z$ of the liquid. If vertical lines be drawn from every point of the bounding line of A to cut the free surface, (or that surface produced if necessary), the volume of the prism thus enclosed is $A.z$. Hence the pressure on A may be also stated as equivalent to the weight of the *superincumbent* liquid. By superincumbent liquid is of course meant the vertical prism of liquid extending to the free surface, supposing the liquid extended on all sides by enlarging the vessel.

45. As an example, let a hollow cone ABC, vertex upwards, be filled with water and inverted upon a horizontal table. Here the free surface is a horizontal plane passing through the vertex C. So that if h be the length of the axis, and r = radius of the base,



the pressure on the table

$$= g\rho . \pi r^2 h$$

$$= \text{weight of the volume } ABDE \text{ of water.}$$

The weight of the liquid in the cone $=\frac{1}{3}g\rho.\pi r^2h$.

$\therefore$ pressure on the table : weight of the liquid :: 3 : 1.

Here the form of the vessel causes a smaller quantity of fluid to produce a larger pressure. (*Compare hydrostatical paradox*, *Art.* 18).

46. *Def.* The *whole pressure* of a fluid upon a surface in contact with it, is the *arithmetical* sum of the normal pressures on the elementary areas composing that surface. It is in fact, the *total* pressure which the surface sustains from the action of the fluid.

The whole pressure upon a surface must not be confounded with the *resultant* pressure on the same. The resultant pressure, like resultant force in Statics, is the single pressure which produces the same dynamical effect as all the components taken together. For instance if a body be pressed from three sides by three equal forces each equal to P lbs, the resultant force may be zero, *i. e.* the dynamical effect may be nothing. But the body nevertheless sustains a total pressure of $3P$ lbs. In Hydrostatics, the calculation of the whole pressure is of importance for various purposes.

In the case of a plane surface, the normal pressures are all in parallel directions; hence the magnitude of the resultant pressure will be the same as that of the whole pressure. In curved surfaces, however, the normal pressures are in various directions, and the whole pressure and the resultant pressure will therefore be different.

47. To find the whole pressure of a liquid on a surface in contact with it.

Let A be the area of the surface. Divide it into a very large number of elementary small areas $a_1\ a_2\ a_3$.........&c, so that each of them may be regarded as plane, and upon which the pressure is sensibly uniform; and let z_1, z_2, z_3, &c. be the depths of these areas (*i. e.* their centers of gravity, or centers of inertia) below the free surface of the liquid. Also let ρ be the density of the liquid.

Now the pressure on $a_1 = g\rho\ z_1\ a_1$,

The pressure on $a_2 = g\ \rho\ z_2\ a_2$, and so on.

$\therefore$ The *whole pressure* on $A = g\rho\ (a_1\ z_1 + a_2\ z_2 + \text{.........})$

But if $\bar{z}$ be the depth of the $C.\ G$ of A below the free surface, we have from Statics,

$$A\bar{z} = a_1z_1 + a_2z_2 + \text{......}$$

$\therefore$ The whole pressure on $A = g\rho A\bar{z}$.

Now $A\bar{z}$ denotes the volume of a prism whose sectional area is A, and whose length of axis is $\bar{z}$.

Hence, the whole pressure on a surface is equivalent to the weight of a prism of the liquid, whose base is equal to the area of the surface, and whose height is equal to the depth of its *C. G.* below the free surface of the liquid. In other words, the whole pressure on a surface is equivalent to the pressure on a horizontal plane of the same area drawn through the *C. G.* of the surface.

48. We have supposed that the surface of the liquid is not exposed to any pressure. If however it be subject to a pressure p per unit of area, this pressure p will be transmitted to every point in the area, and in that case the whole pressure on $A = Ap + g\rho A\cdot\bar{z}$.

49 The above formulæ determine the whole pressure when the depth of the *C. G.* of the immersed surface is known. But when, as in more complex cases, this is not known, or the surface can not be divided into a finite number of known areas whose *Cs G.* are known, the evaluation of the expression $a_1z_1 + a_2z_2 + a_3z_3 + \&c.$ must be effected by means of the Integral Calculus.

50. We will apply the formulæ of the preceding articles to calculate the whole pressure in some particular cases.

Example 1. A rectangle is immersed in a liquid of density ρ, with its two sides horizontal, and its plane inclined at an angle θ to the vertical. To find the whole pressure on it.

Let a and b be the sides, and c the depth of the upper side.

$$\text{Here} \quad \bar{z} = c + \frac{b}{2}\cos\theta$$

$$\therefore \text{ whole pressure on the rectangle} = g\rho.ab\left(c + \frac{b}{2}\cos\theta\right).$$

Example 2. A hemispherical bowl is filled with a liquid. To find the whole pressure on its inner surface.

Let r be the radius of the inner surface,

$$\text{then } \bar{z} = \frac{r}{2}, \text{ and the area of the inner surface} = 2\pi r^2.$$

$$\therefore \text{ whole pressure} = g\rho.\pi r^3$$

$\therefore$ whole pressure : weight of the liquid $:: g\rho\pi r^3 : \frac{2}{3}g\rho\pi r^3$ $:: 3 : 2.$

Example 3. An upright hollow cylinder is filled with equal volumes of two different liquids. Compare the whole pressures on the upper and lower halves of the curved surface.

Let ρ and ρ' be the densities of the upper and lower liquids respectively, whose common surface divides the curved surface of the cylinder into two equal portions. Let r = radius of the base, and $2h$ the height of the cylinder.

The whole pressure on the upper half $= g\rho . 2\pi rh . \frac{h}{2} = g\rho\pi rh^2$.

For the whole pressure on lower half, we must remember that every point of the common surface is, in consequence of the liquid ρ, subject to a pressure of $g\rho . h$, and this pressure is transmitted to every point of the lower half of the curved surface. Hence the *whole pressure* on the lower half

$$= g\rho h . 2\pi rh + g\rho' . 2\pi rh . \frac{h}{2} = g\pi rh^2 (2\rho + \rho').$$

Example 4. To find the whole pressure which a diver sustains, when the *C. G.* of the surface of his body is 32 feet under water.

The surface of a middle sized human body may be taken to be about 10 square feet. Hence the pressure on his body expressed in kinetic units of weight $= g\rho 10 \times 32$.

Expressed in lbs., it is nearly

$$10 \times 32 \times \frac{1000}{16} \text{ lbs. or } 20{,}000 \text{ lbs.}$$

EXERCISES.

(1) The upper side of a sluice gate is $10\frac{1}{2}$ feet beneath the surface of water, its dimensions are 3 feet vertical by 18 inches horizontal; calculate the pressure on it. Ans. 3375 lbs.

(2) Find the whole pressure on one side of an equilateral triangular lamina whose side is a ft., if the vertex be b ft. below the surface, its base being horizontal and c ft. below the surface. Ans. $g\rho\frac{a^2}{4\sqrt{3}}(b+2c)$.

(3) An upright cylinder is partly filled with water, and afterwards a heavy body is lowered and allowed to float in it. Show that the whole pressure on the base is increased by the

weight of the body. Show also that if the whole pressure on the base be doubled, the whole pressure on the side will be quadrupled.

(4) A vertical cylinder contains equal volumes of two liquids; d is the density of the upper liquid and W is the whole pressure on the curved surface of the cylinder occupied by it; d' and W' are the corresponding density and whole pressure on the curved surface occupied by the lower liquid. Prove that

$$\frac{w'}{w} - \frac{d'}{d} = 2.$$

(5) A square reservoir of water, 25 feet deep, 100 feet long and 50 feet broad at the surface of the water, has two opposite faces inclined at an angle of 75° to the horizon, and the other two faces are vertical; find the total pressure on the several faces.

Ans. $31250(\sqrt{6} - \sqrt{2})$; $31250 \times \dfrac{\sqrt{3}+1}{3}$.

(6) In a canal lock the water rises to the height of 15 feet against a gate 8 feet broad. Find the whole pressure on the gate. Ans. 26250 lbs.

(7) A vertical cylinder contains equal volumes of two liquids; compare their densities, when the whole pressures on the portions of curved surface in contact with them are in the ratio of 1 : 15. Ans. 1 : 13.

(8) A cube is filled with liquid; compare the pressures on the base and a side. Ans. 2 : 1.

CHAPTER V.

Resultant Pressures on Immersed Surfaces.

51. When the surface of a body is exposed to fluid pressure, every element of the surface, (*i. e.* every minute portion into which it can be divided) is subject to a pressure in the direction normal to it at that point. If it is the case of a heavy liquid at rest, the pressure depends upon the depth of the point below the free surface.

52. The *resultant pressure* on a surface is defined to be the single pressure equivalent in dynamical effect to all these pressures conjointly. Hence the determination of the resultant pressure is equivalent to the determination of the resultant of a system of forces acting at different points of a rigid body.

The point where the direction of the resultant pressure cuts the given surface, or in other words, the point of application of the resultant pressure, where any such exists, is called the *center of pressure* of the surface.

It must be remembered that a single resultant of such a system of forces is not always possible; for instance when the forces are such as to be equivalent to a system of couples. (*see Statics*).

53. When the surface exposed is a plane area, the resultant pressure on it is the same as the *whole pressure* on the area, and is in *magnitude* the same as when the area is turned about its center of inertia into a horizontal position (*see Art* 47.)

In this case, the position of the center of pressure can be determined in the following manner. Let A be the area of the given plane immersed in a liquid at an inclination θ to the vertical line. Let CD be the line of intersection of the plane (or plane produced) with the horizontal free surface of the liquid.

Divide the plane A into thin horizontal strips by lines parallel to CD. The breadth of the strips being very small, the pressure on each may be regarded uniform throughout.

Let $a_1, a_2, a_3, \ldots\ldots$ be the areas of the strips, and z_1, z_2, z_3 &c. be the perpendicular distances from CD of their centers of inertia.

The pressures on the strips are therefore respectively $= g\rho a_1 z_1 \cos\theta,\ g\rho a_2 z_2 \cos\theta,\ g\rho a_3 z_3 \cos\theta$, and so on.

Hence we have to find the center of a system of parallel forces whose magnitudes are known, and the perpendicular distances from CD of whose points of application are,

$$z_1, z_2, z_3 \text{ \&c.}$$

$\therefore$ if $\bar{z}$ represent the distance of the center of pressure from CD, we have by Statics,

$$\bar{z} = \frac{(g\rho a_1 z_1^2 + g\rho a_2 z_2^2 + g\rho a_3 z_3^2 + \ldots\ldots\ldots)\cos\theta}{(g\rho a_1 z_1 + g\rho_2 z_2 + g\rho a_3 z_3 + \ldots\ldots\ldots)\cos\theta} = \frac{\Sigma(az^2)}{\Sigma(az)}.$$

Similarly if y_1, y_2, y_3 &c. be the distances of the centers of inertia of the strips from another line in the plane, perpendicular to CD, and $\bar{y}$ the distance of the center of pressure from that line,

$$\bar{y} = \frac{(g\rho a_1 z_1 y_1 + g\rho a_2 z_2 y_2 + \ldots\ldots\ldots)\cos\theta}{(g\rho a_1 z_1 + g\rho a_2 z_2 + \ldots\ldots\ldots)\cos\theta} = \frac{\Sigma(azy)}{\Sigma(az)}.$$

3

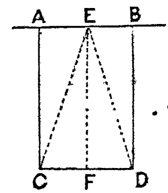
$\bar{z}$ and $\bar{y}$ determine the position of the center of pressure; its vertical depth below the free surface is $\bar{z}\cos\theta$.

Obs. 1. The position of the center of pressure with respect to CD is independent of the inclination of the plane.

Obs. 2. Except in very simple cases, the evaluation of the quantities $\Sigma(az)$, $\Sigma(az^2)$, and $\Sigma(azy)$ can be effected only by means of the Integral Calculus.

54. We will illustrate the method by two very simple applications;—

Example 1. A rectangle is immersed in a liquid with one side coincident with the free surface. To find the center of pressure.



Let a and b the sides of the rectangle, and θ its inclination to the vertical. Divide the length b into n horizontal strips each of breadth $\frac{b}{n}$.

When the breadth of each strip is indefinitely diminished, the pressure on each strip will be sensibly uniform throughout.

The pressure on the r^{th} strip from the top

$=g\rho \cdot a\frac{b}{n}\cdot\frac{br}{n}\cos\theta$, $\frac{br}{n}\cos\theta$ being the depth of its center of inertia below the free surface of the liquid.

Let $\bar{z}$ be the distance of the center of pressure from AB measured along EF, the axis of the rectangle. It is evident from the symmetry of the figure, that the center of pressure lies in this line.

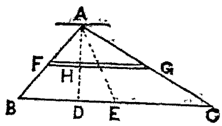
$$\text{Hence, } \bar{z} = \frac{g\rho\frac{ab^3}{n^3}\cos\theta(1^2+2^2+3^2+\dots\dots+n^2)}{g\rho\frac{ab^2}{n^2}\cos\theta(1+2+3+\dots\dots+n)}$$

$$=\frac{b}{n}\cdot\frac{\frac{n(n+1)(2n+1)}{6}}{\frac{n(n+1)}{2}}=\frac{b}{n}\cdot\frac{2n+1}{3}=\frac{b}{3}\left(2+\frac{1}{n}\right)$$

$$=\frac{2b}{3}, \text{ when } n \text{ is indefinitely increased.}$$

This shews that the center of pressure coincides with the $C.G$ of the isosceles triangle CED.

Example 2. A triangle is immersed in a liquid with its base horizontal and vertex in the free surface. To find the depth of the center of pressure.



Let the triangle be divided into n horizontal strips, which may be ultimately considered as rectangles when n is indefinitely increased. Let the height AD of the triangle be h. Then the breadth of each strip $=\frac{h}{n}$.

The pressure over each strip being ultimately uniform throughout, the middle points of those strips will be the points of action of the pressures ; so that the center of pressure will clearly lie in the line AE which bisects BC. Let $BC=a$.

Let FG be the r^{th} strip from the top, and let the perpendicular AD cut FG in H.

$$\text{Then } \frac{FG}{BC}=\frac{AH}{AD}=\frac{r}{n}, \quad \therefore FG=\frac{ra}{n}.$$

$\therefore$ the area of $FG=\frac{ra}{n}\cdot\frac{h}{n}=\frac{rah}{n^2}$; and the depth of its $C.G.$

$=AH\cos\theta=\frac{hr}{n}\cos\theta$, where θ is the inclination of the lamina to the vertical.

$$\therefore \text{ the pressure on the strip } FG=g\rho\cdot\frac{ra}{n}\cdot\frac{h}{n}\cdot\frac{hr}{n}\cos\theta$$

$$=g\rho\frac{ah^2}{n^3}r^2\cos\theta.$$

$\therefore$ if $\bar{z}$ be the distance of the center of pressure parallel to AD,

$$\bar{z}=\frac{\Sigma\left(g\rho\cdot\frac{ah^3}{n^4}r^3\cos\theta\right)}{\Sigma\left(g\rho\frac{ah^2}{n^3}r^2\cos\theta\right)}=\frac{h}{n}\cdot\frac{1^3+2^3+3^3+\dots+n^3}{1^2+2^2+3^2+\dots+n^2}$$

$$=\frac{h}{n}\cdot\frac{\frac{n^2(n+1)^2}{4}}{\frac{n(n+1)(2n+1)}{6}}=\frac{3h}{2}\cdot\frac{(n+1)}{2n+1}=\frac{3h}{2}\cdot\frac{1+\frac{1}{n}}{2+\frac{1}{n}}$$

$=\frac{3}{4}h$ ultimately, when n is indefinitely increased.

It may be shewn in the same manner that when the triangle is immersed with its base in the surface, the distance of the center of pressure measured perpendicular to the base is $\frac{h}{2}$.

55. The determination of the resultant pressure and the center of pressure of a curved surface is more complex, in as much as the normal pressures at different points are differently directed. Moreover in this case, these pressures may not be equivalent to a single resultant.

The general principle of the determination of the resultant pressure in magnitude and direction is the same as that of a system of forces acting upon a rigid body in different directions. The surface having been divided into a large number of very small plane areas, the pressure upon each is resolved into three components parallel to three co-ordinate planes, and the resultant is then found by combining the three systems of parallel pressures by the rules of statics. But this method supposes that the separate normal pressures and their directions are known with reference to these planes, and that the summation can be effected; subjects which in general require the application of the Integral Calculus.

56. For elementary purposes, however, the following method will be found useful in solving the simple cases of resultant pressure upon a surface in contact with a homogeneous heavy liquid at rest.

The resultant pressure on the immersed surface can be resolved into three components, one vertical, and two others parallel to two horizontal lines at right angles to each other.

These components, when separately known, determine the resultant.

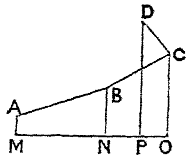
57. In order to avoid circumlocution, it will be convenient to give the following lemma with respect to the orthogonal projection of a surface over a plane.

The *projection* of a line AB upon a given plane is the distance between AN and BN drawn from the extremities of the given line upon the plane.

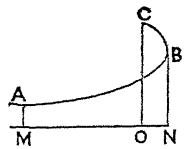
A projection is considered positive when measured in one direction, and negative when measured in the opposite direction.

The projection of a broken line, (the parts lying in the same plane) upon a plane perpendicular to its plane, is the algebraical sum of the projections of its parts.

Thus the projection of the broken line $ABCD$ upon a plane perpendicular to its plane, is MP, which is equal to $MN + NO - OP$; here OP is negative being measured in the opposite direction to MN, *i. e.* from O towards P.



The projection of a plane curve ABC upon a plane perpendicular to its plane, is similarly the line MO, the projection of AB being MN, and that of BC being NO.



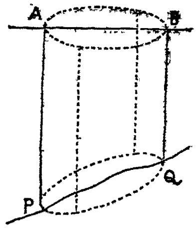
It follows then from the above that the projection of a plane closed curve upon a plane perpendicular to its plane is, algebraically speaking, nothing; for, if the point C coincides with A, O coincides with M, and OM becomes zero. Thus the projection of a horizontal closed curve upon a vertical plane is zero.

The projection of a surface (plane or curved) upon a given plane, is the area on that plane included between the feet of the perpendiculars drawn from every point of the bounding line of the surface, upon the given plane.

Thus let PQ be any surface (plane or curved) and AB the plane upon which it is to be projected; perpendiculars are drawn from every point of the bounding line of PQ, upon the plane AB. The feet of these perpendiculars enclose an area in the plane AB, which is called the projection of PQ on AB.

It may be noticed that the area of the surface has nothing to do with its projection. Different surfaces having the same bounding line will have the same projection.

It follows from this, that the projection of a closed surface, as that of a solid body, upon any plane whatever is, algebraically considered, zero. For taking larger and larger portion of the surface, we see that the bounding line diminishes in extent, so that when the bounding line of the whole surface is ultimately reduced to a point, its projection is of course nothing. In other words, dividing the closed surface into two parts by any plane, the projection of one part is equal in magnitude to that of the other part, but being traced in opposite directions, the algebraical sum is nothing.



58. **To find the vertical component of the resultant pressure on a given surface PQ, in contact with a liquid at rest.**

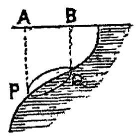
Through the boundary line of the surface, draw vertical lines to meet the free surface in the curve AB, thus enclosing a mass PB of the liquid; consider this mass to be solidified.

The equilibrium of the mass PB is maintained by the horizontal pressures of the surrounding liquid on its curved surface, and by the reaction of PQ, (which is equal and opposite to the resultant pressure of the liquid upon PQ). Hence if R_1 represent the vertical component of the Resultant pressure of the fluid on $P\,Q$, we have

R_1 = weight of the mass PB, and the line of action of R_1 passes through the $C.\ G.$ of PB.

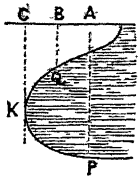
The vertical component of the Resultant pressure is for brevity called the resultant *vertical pressure.* In the figure, R_1 is vertically downwards, being opposite to the resolved component of the reaction of PQ on the liquid. Moreover, the reasoning is the same, if the vertical lines from the boundary line of PQ do not all lie entirely in the liquid, for the pressure at any point of PQ depends only upon the depth of that point below the free surface of the liquid (*see* Art. 33).

Hence in all such cases the resultant vertical pressure on a surface is equivalent to the weight of the superincumbent liquid.



59. When the surface $P\,Q$ is pressed upwards by the liquid as in the annexed figure, R_1 will still be equivalent to the weight of the supposed superincumbent liquid $P\,B$; but the direction of R_1 will be vertically upwards. For suppose the liquid to be extended so as to fill the space PB, and the liquid from the inside removed. The pressure at any point of PQ is now the same in magnitude as before, but reversed in direction, so that R_1 is equal in magnitude to the vertical component of the resultant pressure on PQ in this hypothetical case, but opposite in direction. As however this latter is equivalent to the weight of the supposed column PB of the liquid, R_1 is also equivalent in magnitude to the same, and its direction is upwards.

If the surface PQ be pressed partly upwards and partly downwards, as in the accompanying figure, divide the surface by means of vertical tangent planes into parts as PK, KQ, so that each part may be pressed in the same way throughout; and consider the vertical pressure on each part separately; thus



The resultant vertical pressure on PQ = resultant vertical pressure on PK downwards + *that on* KQ upwards = weight of liquid $PC \sim$ weight of liquid QC.

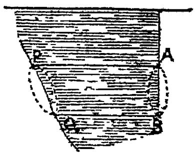
The direction of R_1 will be upwards or downwards according as PC is greater or less than QC.

60. It will be remarked that the reasoning of the preceding articles gives the vertical pressures of even heterogeneous heavy fluids, *i. e.* where gravity is the only impressed force. Such a fluid will necessarily arrange itself in horizontal strata of density varying with the depth, and the weight of the superincumbent fluid has to be determined from its volume and law of density. But in such hypothetical columns of fluid as are considered in Art. 59, the hypothetical extension of the fluid must follow the same law of density.

61. To find the component of the Resultant pressure in a given horizontal direction, on a surface PQ in contact with a fluid.

The horizontal component of the Resultant pressure is, for brevity, called the *horizontal pressure.*

Project the surface PQ on the vertical plane perpendicular to the given horizontal direction; and suppose AB to be the projection.



The equilibrium of the cylinder PB (supposed solidified) is maintained by

(1) The pressures of the surrounding liquid on its curved surface, which are all perpendicular to the axis of the cylinder.

(2) The re-action of the surface PQ on the cylinder, which is equivalent to the resultant pressure of the liquid on PQ in magnitude, but opposite in direction.

(3) The pressure of the surrounding liquid on the plane area AB of the cylinder, which is equal in magnitude to the whole

pressure of the liquid on AB, and will be known by Art. 47, if the nature of AB be known.

(4) The weight of PB, which is vertically downwards.

Hence if R_2 denote the horizontal pressure on PQ in the given direction, we have for equilibrium,

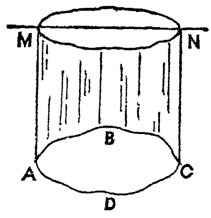
R_2 = Resultant pressure (or Whole pressure) on AB, and the direction of R_2 is opposite to the direction of this resultant pressure, and passes through the center of pressure of AB.

62. In a similar way, the component of the resultant pressure in another horizontal direction at right angles to the former, may be determined; and these two horizontal components, together with the vertical component, may in certain cases be compounded into a resultant by the rules of Statics. Thus if R_1, R_2 and R_3 be the components of the resultant pressure R, then $R^2 = R_1^2 + R_2^2 + R_3^2$. The conditions for the existence of the resultant may be discussed as in Statics. Also if α, β and γ be the inclinations of the resultant R to the directions of R_1, R_2 and R_3, we have

$\cos\alpha = \frac{R_1}{R}$, $\cos\beta = \frac{R_2}{R}$, and $\cos\gamma = \frac{R_3}{R}$, which determine the direction of the resultant.

63. We will illustrate the subject in certain special cases.

Example 1. To find the resultant pressure on the surface of a solid body wholly immersed in a liquid.



The surface of a solid body is a closed surface, and has no projection on any plane whatever. Hence the resultant horizontal pressure in any direction is zero. In other words, the horizontal pressure on one part of the surface is counteracted by that on the other. The resultant pressure on the surface is therefore entirely vertical. For the vertical pressure, draw vertical tangent planes so as to divide the surface into parts as ABC and ADC, upon which the vertical pressures are downwards and upwards respectively.

The vertical pressure on ADC is equivalent to the weight of the superincumbent liquid $ADCNM$, and is upwards; and that on ABC is

$=$ weight of the superincumbent liquid $ABCNM$, and is downwards.

$\therefore$ vertical pressure on the whole surface
$=$ weight of liquid $ADCNM -$ weight of liquid $ABCNM$
$=$ weight of liquid $ADCB$.

Thus the vertical pressure on the whole surface of an immersed body is upwards, and is equal in magnitude to the weight of the liquid displaced by the body. Also the line of action of this pressure passes through the $C.G$ of the displaced fluid.

Ex. 2. To find the Resultant pressure on the surface of a partly immersed body.

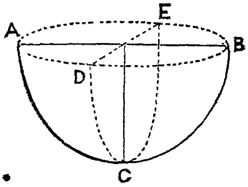
The boundary line of the surface immersed is, in this case, the curve of section of the body by the horizontal free surface of the liquid.

The projection of the immersed surface on a vertical plane is the same as that of the area of the section, and is therefore nothing, (*See Art.* 57). Hence the Resultant horizontal pressure in any direction whatever is nothing.

The vertical component, which is therefore the whole resultant, is equivalent to the weight of a column of the liquid displaced by the body.

Also the center of pressure is the point, where the vertical line through the $C. G.$ of the displaced liquid cuts the surface immersed.

Ex. 3. A hemispherical bowl with axis vertical is filled with liquid. To find the Resultant pressure on a fourth portion of its surface cut off by two vertical planes through its center at right angles to each other.



Let ADC be one of the four parts of the surface, r being the radius of the bowl.

The projection of ADC on each of the vertical planes of section is a quadrant of a circle bounded by a horizontal and a vertical radius of the bowl, the depth of the $C.G$ of the quadrant below the free surface of the liquid being $\frac{4r}{3\pi}$.

$\therefore$ Resultant horizontal pressures perpendicular to these planes, are each equal to

$$\tfrac{1}{4}g\rho \cdot \pi r^2 \cdot \frac{4r}{3\pi} = \tfrac{1}{3}g\rho r^3 ;$$

and the vertical resultant pressure being the weight of the superincumbent liquid,

$$=g\rho\cdot\tfrac{1}{8}\cdot\tfrac{4}{3}\,\pi r^3=\tfrac{1}{6}g\rho\cdot\pi r^3.$$

$$\therefore \text{ Resultant pressure}=g\rho r^3\sqrt{\left(\tfrac{1}{9}+\tfrac{1}{9}+\frac{\pi^2}{36}\right)}=\tfrac{1}{6}g\rho r^3\sqrt{(8+\pi^2)}$$

Also, since the normal pressure at every point of the bowl passes through its center, the direction of the resultant pressure is that of a radius, the cosines of whose inclinations to the given horizontal and vertical lines are respectively

$$\frac{2}{\sqrt{(8+\pi^2)}},\ \frac{2}{\sqrt{(8+\pi^2)}}\text{ and }\frac{1}{\sqrt{(8+\pi^2)}}$$

The center of pressure is the point where this line cuts the surface of the bowl.

Ex. 4. A conical vessel ABC, with vertical axis is filled with liquid. To find the resultant pressure on either half ($DECA$) of the surface, when divided by a vertical plane through the axis.

Let r = radius of the base, and h = height of the cone. The resultant vertical pressure on $ACDE$ = weight of the superincumbent liquid $=\frac{1}{2}\cdot\frac{1}{3}\,g\rho\pi r^2h.$

Again, the projection of the surface $ACDE$ on the vertical plane of section is the triangle CDE.

$\therefore$ Resultant horizontal pressure perpendicular to the section

CDE = resultant pressure on $CDE=g\rho rh\cdot\dfrac{h}{3}=\frac{1}{3}g\rho rh^2.$

The resultant horizontal pressure at right angles to the above is evidently nothing, for the surface $ACDE$ has no projection on a vertical plane at right angles to the plane of section ;

$$\therefore \text{ the resultant pressure}=\sqrt{\left\{\left(g\frac{\rho\pi r^2h}{6}\right)^2+\left(g\frac{\rho rh^2}{3}\right)^2\right\}}$$
$$=\tfrac{1}{6}g\rho rh\sqrt{(\pi^2r^2+4h^2)}.$$

The direction of the resultant pressure evidently passes through some point in the axis. If θ be the angle which it makes with the axis,

$$\tan\theta=\frac{\frac{1}{3}rh^2}{\frac{1}{6}\pi r^2h}=\frac{2h}{r\pi}.$$

EXERCISES.

(1) Shew that if a body be depressed in a fluid, the whole pressure on its surface will be increased, but the resultant pressure will remain unaltered.

(2) A rectangular box has its lid in the surface of a liquid. Find the resultant horizontal pressures on any two adjacent sides.

(3) A vertical solid cylinder of radius r and length h is immersed in water, with its upper end at a depth a below the surface; explain what is meant by the total normal pressure upon it and by the resultant pressure; and determine each.
Ans. (1) $w\pi r(h+r)(h+2a)$. (2) $w\pi hr^2$.

(4) A hemispherical bowl with a small hole at the end of its axis is placed with its rim downwards on a horizontal table. Find the weight of the bowl if the water begins to flow under the rim the moment the bowl becomes filled by pouring water through the hole. Ans. $\frac{1}{3}w\pi r^3$.

(5) Find the number of ounces in (1) the total normal pressure, and (2) in the resultant pressure, on the surface of a sphere whose radius is one foot, and whose center is 10 feet below the surface of a tank, assuming that the volume of a sphere whose radius $= r$ feet is $\frac{88}{21}r^3$ cubic feet, the area of its whole surface $\frac{88}{7}r^2$ square feet, and the weight of one cubic foot of water 1000 oz.? Ans. (1) 7857·14 lbs. (2) 261·9 lbs.

(6) Find the horizontal component, in tons, of the fluid pressure on a flood gate 10 feet long and 20 feet wide, inclined to the vertical at an angle of 30°. Ans. 20·927 nearly.

(7) A right-angled cone is immersed with its axis vertical, and its vertex in the surface of a uniform fluid at rest; compare the total pressure on the whole surface, with that on the curved surface, and with the resultant pressure. Ans. $3+2\sqrt{2} : 2 : 1$.

(8) A sphere is immersed in water, and its centre is at a depth of 18 feet below the surface; the whole pressure on the sphere is g times the resultant vertical pressure; find the radius of the sphere. Ans. 6 ft.

(9) A closed cubical vessel filled with water stands on a horizontal plane so that a face F is vertical; it is then tilted about one edge of its base so that a diagonal of the face F is vertical; compare the pressure upon the face F in the two positions. Ans. $1 : \sqrt{2}$.

(10) A sphere 6 feet in diameter is just filled with water. Find the total and resultant pressures on the upper and lower halves of the spherical surface.

(11) Find the depth of the center of pressure of a triangle just immersed with its base in the surface of a heavy incompressible liquid. Ans. Half the depth of the vertex.

(12) If the difference between the pressures on the ends of a solid cylinder completely immersed in a fluid be to the resultant pressure on the curved surface as $\sqrt{(3)} : 1$, shew that the axis of the cylinder is inclined at an angle of 30° to the vertical.

(13) The water in a canal lock rises to a height of 16 feet; calculate the pressure on a sluice gate 3 feet high by 4 feet wide, having its upper edge 4 feet from the surface of the water.

Ans. 4125 lbs.

(14) A rectangular sluice gate, measuring 5 feet in width and 10 feet in depth is supported by the sides of a canal; if the water be level with the top on one side and reach half-way upon the other, find the pressure of the gate on the sides of the canal.

Ans. $11718\frac{3}{4}$ lbs.

(15) A river wall, 200 yards long is built in courses of masonry of one foot high; the water rises against it to a height of 6 fathoms; calculate the pressures against the 1st, 18th, and the 36th courses. Ans. $300w$, $10500w$, $21300w$.

(16) Water rises to different heights h and h' at each side of a vertical rectangular flood gate, which can turn round its base whose length is a. Prove that the horizontal force P, applied perpendicularly to the gate at a height H from the base, which will prevent its turning is

$$P = \frac{wa}{6H}(h^3 - h'^3).$$

(17) A rectangular trough of length b and height c is filled with water, and then tilted at a given angle θ, so that only one edge of its base remains in contact with the ground. Find the resultant vertical pressure upon the side bounded by that edge. For what inclination will it be greatest?

Ans. (1) $\frac{1}{4}bc^2 \sin 2\theta$. (2) 45°.

(18) A cubical box, filled with water, has a close fitting heavy lid fixed by small hinges to one edge; compare the tangents of the angles through which the box must be tilted about the several edges of its base, in order that the water may just begin to escape. Ans. 3 : 6 : 4.

(19) A rectangular dock gate is 30 feet deep and 11 feet broad; find in tons the pressure upon it when the dock is full.

Ans. 138·1 nearly.

(20) If a quadrilateral lamina $ABCD$, in which AB is parallel to CD, be immersed in liquid with the side AB in the surface, the center of pressure will be at the point of intersection of AC and BD, if $AB^2 = 3CD^2$.

CHAPTER VI.

On Floating Bodies.

64. To find the conditions of equilibrium of a body freely floating in a fluid at rest.

In such a case the body is acted on by its own weight downwards, and by the resultant fluid pressure on its immersed surface, which (by Art. 63, examples 1 and 2) is equivalent to the weight of the displaced fluid, and acts vertically upwards through the $C.\ G.$ of the displaced fluid.

Hence for equilibrium of the body, we must have (*a*) weight of the body = weight of the fluid displaced by it, and (*b*) the weight of the body and the resultant fluid pressure on its surface must be oppositely directed in the same straight line; in other words, the centers of gravity of the body and the fluid displaced must be in the same vertical line. This latter condition gives the position of equilibrium of the body.

65. Let V be the volume of the body, and V' that of the part immersed; ρ and ρ' the densities of the body and the fluid respectively, both being homogeneous.

Then $g\rho V = g\rho' V'$, or $V : V' :: \rho' : \rho$; or the volume of the part immersed is inversely proportional to the density of the fluid. Thus, if the body float in a liquid whose density is twice that of the body, half of its volume will be immersed.

When the solid floats completely immersed in a liquid, $V = V'$ and therefore $\rho = \rho'$. Hence a homogeneous solid will float completely immersed, provided it is of the same density as the liquid. The centers of gravity of the solid and the liquid in this case coincide. A heterogeneous body is also capable of so

floating, provided the weight of the body be equal to the weight of the liquid displaced by it, and the centers of gravity of the body and the liquid displaced be in the same vertical line.

66 When a solid floats partially immersed in different liquids, the resultant pressure on its surface is still vertically upwards, and is equivalent to the sum of the weights of the liquids displaced by the body. In other words, the whole displaced fluid must be supposed to consist of strata of the same kind as, and continuous with, the horizontal strata of the surrounding fluid. For instance, when a body floats partially immersed in water, the weight of the body must be equal to the weight of the water displaced together with that of air displaced. In the hypothetical case of a balloon floating freely in uniform air at rest, the condition of equilibrium is that the weight of the balloon be equal to that of the air displaced.

67. DEF. The section of the floating body by the horizontal free surface of the liquid is called the *plane of floatation*. If the floating body be a plane lamina, the plane of floatation becomes the *line of floatation*.

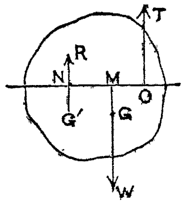
The resultant vertical pressure on the immersed surface of the body, which is equal to the weight of the fluid displaced, is called the *buoyancy* of the body; and the C. G. of the fluid displaced, through which the resultant vertical pressure acts, is called the *center of buoyancy*.

68. The conditions of equilibrium in Art. 64 may accordingly be stated thus:

The weight of the body must be equal to its buoyancy, and the C. G. of the body must lie in the vertical line through the center of buoyancy.

The maximum buoyancy of a body occurs, of course, when it is completely immersed. Thus if V be the volume of the body, and S and S' the specific gravities of the body and the liquid, the maximum buoyancy of the body is VS'. The body is acted on by the forces VS and VS' in opposite directions; hence it will sink down or rise up according as $VS >$ or $< VS'$ *i.e.* as $S >$ or $< S'$. That is, a homogeneous body can not float completely immersed in a liquid heavier than itself. It can of course float partially immersed when the buoyancy has adjusted itself to its weight.

69. To find the conditions of equilibrium of a floating body, partly supported at a point.



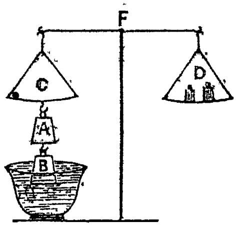
Let W be the weight of the body, R that of the fluid displaced, and T the force of constraint at the given point.

Now since the body is at rest under the action of the forces W, R and T, of which W and R are vertical, therefore the force T must also be vertical and must lie in the plane of the forces W and R. Let the directions of the forces meet the plane of floatation at M, N and O. Then for equilibrium of the body we have

$$T = W \sim R \ldots\ldots\ldots (1).$$

and taking moments about O, we have $W \cdot OM = R \cdot ON \ldots\ldots (2)$ Equation (1) shews that T must be upwards or downwards according as W is greater or less than R. Equation (2) determines the position of equilibrium. If W and R be in the same vertical line, T must be also in the same vertical line. This must necessarily be the case when a homogenous body is completely immersed in a uniform liquid ; for then the centers of gravity of the body and the displaced liquid coincide.

70. When a body is completely immersed, T or $W \sim R$ is called the *apparent weight* of the body in the liquid, the *real weight* being its weight in vacuo. In other words, the apparent weight of a body is its real weight diminished by its buoyancy. The weight of the liquid displaced is *the weight lost.* "A body immersed in a liquid loses a part of its weight equal to the weight of the displaced liquid." This is known as the celebrated principle of Archimedes, and may be verified by the following experiment.



A hollow cylinder A is suspended to one pan C of a balance, and under this a solid cylinder B, whose volume is exactly equal to the capacity of A. Equipoise is placed on the other scale pan.

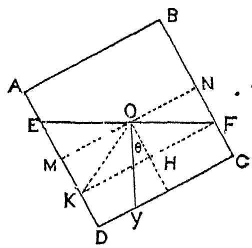
If now the cylinder A be filled with water, the equilibrium is disturbed. But if in this position, a vessel of water be placed beneath so as to immerse the cylinder B in it, the equilibrium will be restored. This shews that by being immersed in water, the cylinder B loses a portion of its weight equal to that of the water in the cylinder A, *i.e.* of the water displaced by itself.

71. We shall illustrate the conditions of equilibrium stated in Arts 64 and 69, by application to some particular examples :

Example 1. To find the positions of equilibrium of a square lamina floating freely with its plane vertical, in a liquid of double its own density.

In order that the first condition may be satisfied, it is necessary that half of the square should be immersed ; or that the line of floatation should pass through the center O of the square.

Again the $C. G.$ of the displaced liquid (in this case, of the portion of the lamina immersed) must be in the vertical line through the center of the lamina. This is clearly satisfied by the lamina floating with either diagonal vertical, or two sides vertical.



To examine whether there is any other position of equilibrium. Let the line of floatation EF make an angle θ with the side AB. Let x be the distance of the C. G. of the immersed area $CDEF$ from the vertical line OY drawn through O.

Draw FK and MON parallel to AB, join KO. Let $AB=2a$. Then $NF=a \tan\theta$, $FC=a(1-\tan\theta)$, and $KO=a\sec\theta$

Also if H be the C. G. of KC,

$$OH=a\tan\theta+\frac{a}{2}(1-\tan\theta)=\frac{a}{2}(1+\tan\theta).$$

The figure $CDEF$ is the sum of the rectangle KC and the triangle FEK. Hence by the rules of Statics,

$$x.\,CDEF=KC.OH\sin\theta-EFK.\ \tfrac{1}{3}OK\sin\left(\frac{\pi}{2}-2\theta\right),$$

$$=a^3\sin\theta(1-\tan^2\theta)-\tfrac{2}{3}a^3\sin\theta\,\frac{\cos 2\theta}{\cos^2\theta}$$

$$=\tfrac{1}{3}a^3\sin\theta(1-\tan^2\theta)$$

$\therefore$ x can not vanish for any value of θ other than 0° or 45°, *i. e.* the C. G. of the displaced fluid cannot be on the vertical line through O, except in the positions we have already enumerated.

Ex. 2. A perfectly spherical balloon (radius $= r$) floats in air, to find the weight it can support.

Let ρ = density of air, and ρ' = density of the gas with which the balloon is filled,

P = the weight of the car and the accessories,

W = weight supported on the car,

w = weight of a unit of area of the envelope.

$\therefore$ The weight of the envelope $= 4\pi r^2 w$, and the buoyancy of the balloon $= g\rho \cdot \frac{4}{3}\pi r^3$.

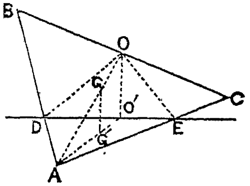
$\therefore$ For equilibrium, we have

$$P + W + g\rho' \tfrac{4}{3}\pi r^3 + 4\pi r^2 w = g\rho \tfrac{4}{3}\pi r^3.$$

$$\therefore \quad W = \tfrac{4}{3} g\pi r^3 (\rho - \rho') - 4\pi r^2 w - P.$$

Note. A silk balloon filled with common coal gas which is capable of supporting 3 men, is about 12 yards in diameter, and when quite full is about 680 cubic yards. To make the balloon rise up, the buoyancy must be a little in excess of the weight.

Ex. 3. A triangular lamina floats in a liquid with one angle immersed, the plane of the triangle being vertical; to determine the position of equilibrium.



Let DE be the line of floatation in the position of equilibrium. Bisect BC and DE in O and O', and let G and G' be the centers of gravity of ABC and ADE respectively. By the condition of equilibrium, $G\,G'$ is vertical.

Also since $\dfrac{AG}{GO} = \dfrac{AG'}{G'O'} = \dfrac{2}{1}$,

$\therefore$ OO' is vertical, and perpendicular to DE. And $DO = OE$.

Let $AD = x$, $AE = y$, $AO = l$, ρ = density of the triangle, and ρ' = that of the liquid.

$\therefore$ By the condition of equilibrium,

$ABC \cdot \rho = ADE \cdot \rho'$, or $\frac{1}{2} xy \sin A \cdot \rho' = \frac{1}{2} b \cdot c \sin A \cdot \rho$.

$$\therefore \quad xy\rho' = b \cdot c \cdot \rho \qquad \text{(1)}$$

Again, $\left.\begin{aligned} DO^2 &= x^2 + l^2 - 2xl \cos BAO \\ OE^2 &= y^2 + l^2 - 2yl \cos CAO \end{aligned}\right\}$

$$\therefore \quad x^2 + l^2 - 2xl \cos BAO = y^2 + l^2 - 2yl \cos CAO \qquad \text{(2)}$$

$$\therefore \quad x^2 - 2xl \cos BAO = y^2 - 2yl \cos CAO.$$

From (1) and (2) we have,

$$x^2 - 2xl\cos BAO = \frac{b^2c^2}{x^2}\left(\frac{\rho}{\rho'}\right)^2 - 2\;\frac{bc}{x}\frac{\rho}{\rho'}\;l\cos CAO.$$

or $x^4 - 2x^3 l\cos BAO + 2b\cdot cl\cdot\frac{\rho}{\rho'}\,.\,x\cos CAO - b^2c^2\left(\frac{\rho}{\rho'}\right)^2 = 0.$

The positive roots of this equation which are less than c determine the positions of equilibrium. If there be no positive root less than c, the triangle cannot float with one angle immersed. Also since the last term is negative, there may be either only one positive root, or three. Hence there may be one or three positions of equilibrium of the triangle.

If the triangle be equilateral, we have

$$\angle BAO = \angle CAO = 30°, \text{ and } a = b = c, \text{ and } l\cos BAO = \tfrac{3}{4}a$$

Hence equation (2) becomes $x^2 - \frac{3}{2}ax = y^2 - \frac{3}{2}ay.$

$$\therefore\; (x-y)(x+y-\tfrac{3}{2}a) = 0$$

If $x+y-\frac{3}{2}a=0$, combining this with (1), we have

$$x + \frac{a^2\rho}{x\rho'} - \tfrac{3}{2}a = 0$$

$$\therefore\; x^2 - \tfrac{3}{2}ax + \frac{a^2\rho}{\rho'} = 0$$

$$\therefore\; x = \frac{a}{4}\left(3 \pm \sqrt{(9 - 16\frac{\rho}{\rho'})}\right)$$

and similarly $y = \frac{a}{4}\left(3 \mp \sqrt{(9-16\frac{\rho}{\rho'})}\right)$

The conditions of x and y both having a possible positive value less than a, are

$$9 > 16\frac{\rho}{\rho'}, \text{ and } 3 + \sqrt{(9 - 16\frac{\rho}{\rho})} < 4$$

$$i.\,e.\ \frac{\rho}{\rho'} \text{ must lie between } \tfrac{9}{16} \text{ and } \tfrac{8}{16}.$$

Ex. 4. Two bodies of different specific gravities balance one another in air; to find the difference of their real weights. Let W and W' be their real weights, V and V' their volumes, and S the sp. gr. of the air. Then since the apparent weights are equal, we have $W - VS = W' - V'S$, or $W - W' = (V - V')S$.

This shews that the *real* weight of the body having the greater volume, *i. e.* having less specific gravity, is greater than that of the other. For instance, a *pound* of cotton is really heavier than a *pound* of lead.

Obs. Except when great accuracy is demanded, the quantity $(V-V')S$ being small, is neglected. In weighing valuable articles as diamonds &c., however, the counterpoises should have as far as possible the same specific gravity as the substance weighed.

72. When a freely floating body is slightly displaced from its position of equilibrium, one of the following three cases is possible ;

(1) The body may tend to return to its position of equilibrium ; in this case, the equilibrium is said to be *stable.*

(2) The body may tend to fall further away from that position ; in this case, the equilibrium is said to be *unstable.*

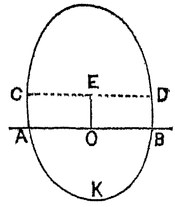
(3) The body may remain in equilibrium in its displaced position ; its equilibrium is then said to be *neutral.*

Displacement is either linear or angular, or both. An arbitrary displacement in general involves both linear and angular displacement. When the displacement is very small, the effects of the two displacements may be separately treated.

73. When a homogeneous body floats in a homogeneous liquid completely immersed, the C. G. of the body and of the displaced liquid coincide. If any displacement, either linear or angular, be given to such a body, it is evident that the buoyancy of the body still acts upwards through the center of gravity of the body, and counteracts the weight. Hence when the center of buoyancy coincides with the C. G. of the body, any position of the body is a position of equilibrium, or the equilibrium is neutral.

74. Suppose a small vertical displacement be given to a freely floating body, partially immersed. It is evident that a depression will increase the buoyancy, and on the other hand, an elevation will diminish it. Hence in either case, the tendency of the fluid pressure is to restore the body to its original position of rest; so that for all vertical displacements, the equilibrium is stable.

It must not be imagined that the body, after being left free, will at once resume its original position of equilibrium. All we can infer is that it will *tend* to move towards that position. As a matter of fact, the body will continue to make vertical oscillations about its position of equilibrium, until the friction of the liquid and other analogous resisting forces gradually destroy the oscillation and bring the body to rest.



For let the floating body be given a small vertical displacement so that the original plane of floatation CD is raised through OE or x, AB being the new plane of floatation. Since the displacement is small, the portion $ABDC$ of the body may be considered cylindrical.

Let the area of the section AB be A, ρ the density of the liquid, and P the force acting downwards upon the body in its new position, m = mass of the body, and a the acceleration of velocity which P produces in m. Then $ma = P$ = weight of the body − weight of the liquid AKB displaced,

= weight of liquid CDK − weight of the liquid AKB.

= weight of liquid $ABDC = g\rho A \cdot x$.

$$\therefore\ a = \frac{g\rho A}{m} \cdot x$$

Hence the acceleration of the body is always towards the point O, and is proportional to the distance of E from that point. Hence the body's motion is harmonic; in other words, it oscillates about the point O, the time of a complete oscillation being

$2\pi\sqrt{\dfrac{m}{g\rho A}}$. (*See Dynamics*).

If V be the volume of the liquid displaced in its position of equilibrium,

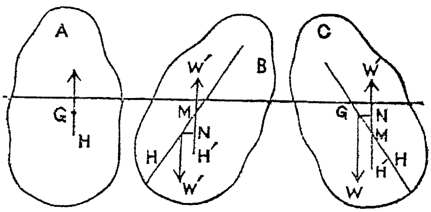
$$mg = g\rho V \text{ or } m = V\rho$$

$$\therefore \text{ time of a complete oscillation} = 2\pi\sqrt{\frac{V}{gA}}.$$

Obs. It must be remarked that the reasoning of Arts. 73 and 74 applies only to *rigid* floating bodies. If the change of pressure on the surface caused by the displacement have the effect of compressing or expanding any portion of the floating body, the equilibrium is not necessarily stable or neutral, but may be unstable. For instance if a hollow cylinder, with bottom open, be immersed in water, the air inside will be gradually compressed, and at a certain depth the cylinder will remain in equilibrium, namely when the weight of the cylinder together with that of the compressed air inside equals the weight of the water displaced. But any vertical displacement either upwards or downwards from that position will, by changing the depth, either expand the air inside or compress it further; and consequently the buoyancy will change, and cause the cylinder to

rise up or sink down according as the displacement is upwards or downwards. In other words, the equilibrium of the cylinder in such a position is *unstable*.

75. We will now consider the case of a very slight angular displacement of a body floating partly immersed.



Let G be the C. G. of the body, and H that of the liquid displaced. Then G and H are in the same vertical line.

The line GH may be regraded as a fixed line in the body, and is called an *axis of floatation*. Now, let the body being displaced through any small angle from its position of equilibrium assume, for the moment, the positions as in figure B or C.

Let H' be the C. G. of the displaced fluid in this position, so that the resultant fluid pressure W' on the body acts upwards through H'. Let the vertical line through H' cut the axis GH in M, which will be either above or below G as in the figures B and C.

If the displacement be indefinitely small, and be about a line in the plane of floatation passing through its center of gravity, the volume of the liquid displaced will remain unchanged. In such a case, the limiting position of M is definite, and is called the *metacenter* of the body. In the displaced position, the body is acted on by two forces, *viz*, the weight W of the body acting vertically downwards through G, and the buoyancy W' acting vertically upwards through H'. Draw GN perpendicular to $H'M$. The resultant moment of the forces W and W' about G is $W'.GN$. It is evident that this moment tends to turn the body in the direction of displacement or opposite to it, according as M is below or above G. Hence the equilibrium of the body is *stable* or *unstable* according as the *metacenter* is above or below the center of gravity of the body.

If the metacenter M coincide with G, the moment $W'.GN$ is zero, and has no tendency to move the body either way. The displaced position is therefore a position of equilibrium, and the equilibrium of the body is *neutral*.

The determination of the metacenter in general can not be effected except by the application of the calculus. It will be seen that its position depends upon the form of the immersed

surface. The stability of equilibrium is greater, the greater the distance of the metacenter from the C. G. of the body. Hence whatever may be the form of the floating body, by judiciously ballasting its lower part so as to bring the C. G. lower down as much as possible, the stability will be in a good measure secured.

Note. When a floating body is turned through a *finite* angle from its position of rest, the moment of the fluid pressure will tend to turn the body back or further away in the direction of displacement, according as the point M, (no longer the metacenter) where the vertical line through the new C. G, of the displaced fluid cuts the axis of floatation, lies above or below G. But it must not be inferred that if M be above G, the body will *return to its original position of rest.* For in displacing the body through a finite angle, we might have displaced it through its other position or positions of equilibrium. For all objects in nature, the positions of *stable* and *unstable* equilibrium occur alternately. It is only when the displacement is through an infinitely small angle, that we may be sure of no other positions of equilibrium having been passed through, and in that case alone the position of M, called the metacenter, being above or below G, will determine the nature of the equilibrium from which the body has been displaced.

76. When the immersed part of the floating body is spherical, the metacenter is easily determined.

Let O be the center of the spherical surface. Then since the fluid pressures at every point of the immersed surface, being along the normal at that point, converge to the center O, the resultant pressure also passes through O, and this is the case when the body is slightly displaced, provided the portion immersed is still spherical. Hence the center O of the spherical surface immersed is the metacenter.

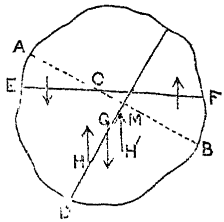
Example. A small iron nail is driven into a wooden sphere, find its positions of equilibrium in water, and examine the nature of the equilibrium.

Let O be the center of the sphere, and N the nail. The center of gravity of the sphere and the nail is therefore a point G in the line ON. By Art. 64, the sphere will float in a position so as to have G in the same vertical line with the C. G. of the displaced water.

Hence it is evident, the sphere can float in only two positions, (1) when N is the highest point, (2) when N is the lowest point. Again, the metacenter of the sphere is the center O, in both positions of equilibrium. In the first position, O is obviously below G, and the equilibrium is unstable, or a slight

angular displacement will upset the equilibrium. In the second position, O is above G, and the equilibrium is stable.

77. *To determine the Metacenter of a plane lamina.*



Let the lamina, floating in equilibrium with the line of floatation AB, be slightly turned in its own plane through an angle θ about C, the middle point of AB. Let EF be the new line of floatation. Since θ is very small, the area AEC will be very nearly equal to the area BCF, each being $\frac{1}{2}AC^2.\theta$.

The volume of the displaced liquid therefore practically remains unchanged. Let A be the area of ADB or EDF, and ρ the specific gravity of the liquid. In the new position, the resultant pressure on the lamina is a force equal to $g\rho.EDF$, acting upwards through H', the C. G. of EDF. Now $g\rho.EDF$ acting upwards through H' is equivalent to the force $g\rho.ADB$ acting upwards through H' + the force $g\rho\cdot BCF$ acting upwards through its C. G. + the force $g\rho\cdot ACE$ acting downwards through its C. G.

Take moments about G of the forces acting on the lamina. The moment of $g\rho\cdot ADB = g\rho\cdot A\cdot GH\cdot\theta$.

The forces $g\rho\cdot BCF$ acting upwards and $g\rho\cdot ACE$ acting downwards form a couple, whose moment about G, being equal to the moment about C, is equal to

$$g\rho\cdot 2\cdot\tfrac{2}{3}\cdot AC\cdot\tfrac{1}{2}AC^2\cdot\theta = g\rho\cdot\tfrac{2}{3}AC^3\theta.$$

$\therefore$ the moment of the resultant pressure about G

$$= (\tfrac{2}{3}AC^3 - A\cdot GH)g\rho\cdot\theta.$$

And it is evident from the direction of the forces that this moment will tend to restore the lamina to its position of equilibrium or turn it further away, according as it is positive or negative ; *i. e.* according as

$$\tfrac{2}{3}AC^3 > \text{ or } < A\cdot GH.$$

Again, let the vertical line through H' cut HG in M, then the moment of the resultant pressure about G is $+g\rho\cdot A\cdot GM\cdot\theta$ or $-g\rho\cdot A\cdot GM\cdot\theta$, according as M is above or below G ; *i. e.* in both cases it is equal to $A(HM - GH)g\rho\cdot\theta$.

$$\therefore\ \tfrac{2}{3}AC^3 - A\cdot GH = A(HM - GH).$$

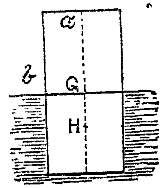
$$\therefore\ \tfrac{2}{3}AC^3 = A\cdot HM. \quad \therefore\ HM = \frac{\frac{2}{3}AC^3}{A},$$

which determines the position of M with respect to H.

$\therefore$ The equilibrium will be stable or unstable according as $A \cdot HM >$ or $< A \cdot HG$. *i. e.* according as M is above or below G

If M coincides with G, the equilibrium is neutral.

78. We will illustrate the above by its application to some simple cases.



Example 1. A rectangle whose sides are $2a$ and $2b$, floats with its plane and two sides vertical, in a liquid of twice its own density. To examine whether the equilibrium is stable or unstable for slight angular displacement in its plane.

Here the line of floatation passes through the C. G. of the lamina.

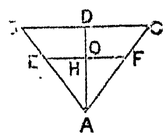
The area of the portion immersed $= 2ab$.

$$\therefore \text{ If } M \text{ be the metacenter, } HM = \frac{\frac{2}{3}a^3}{2ab} = \frac{a^2}{3b}$$

$$\text{and } HG = \tfrac{1}{2}b$$

$$\therefore M \text{ is above or below } G \text{ according as } \tfrac{1}{3}\frac{a^2}{b} > \text{ or } < \tfrac{1}{2}b.$$

$$i.\ e. \text{ as } \frac{a^2}{b^2} > \text{ or } < \tfrac{3}{2}.$$



Example 2. An isosceles triangle floats in a liquid with its base horizontal and above the surface. Find the position and nature of equilibrium.

Let ρ and ρ' be the densities of the lamina and the liquid, $h = AD$ the height, and $x = AO$, the portion of the axis immersed.

We have from the condition of equilibrium,

$$ABC \cdot \rho = AEF \cdot \rho'$$

$$\therefore \frac{\rho}{\rho'} = \frac{AEF}{ABC} = \frac{x^2}{h^2}, \quad \therefore x = h\sqrt{\left(\frac{\rho}{\rho'}\right)}.$$

This determines the position of equilibrium. Also the axis being vertical, and the Cs. G. of the lamina and the fluid displaced being in the axis, the second condition of equilibrium is satisfied. Let $BC = 2a$,

$$\text{Now } \frac{EO}{BD} = \frac{AO}{AD}, \quad \therefore EO = \frac{a.x}{h}.$$

$\therefore$ If M be the metacenter,

$$HM=\tfrac{2}{3}\frac{EO^3}{AEE}=\frac{\tfrac{2}{3}EO^3}{EO.x}=\tfrac{2}{3}\frac{a^2x^2}{h^2x}=\tfrac{2}{3}\frac{a^2x}{h^2}$$

$$\text{and } HG=\tfrac{2}{3}(AD-AO)=\tfrac{2}{3}(h-x)$$

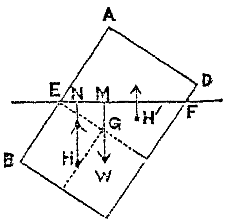
$\therefore$ The equilibrium is stable or unstable, according as

$$\frac{2}{3}\frac{a^2x}{h^2}> \text{ or } <\tfrac{2}{3}(h-x),$$

$$i.\ e.\ \text{as } \frac{a^2}{h^2}> \text{ or } <\frac{h}{x}-1$$

$$i.\ e.\ \text{as } \frac{a^2}{h^2}> \text{ or } <\sqrt{\left(\frac{\rho'}{\rho}\right)}-1.$$

Example 3. A rectangular lamina $ABCD$ rests in a liquid of twice its own density, with two sides vertical; and is moveable in its plane about E, the middle point of one of its vertical sides AB. Find the nature of equilibrium for a small angular displacement θ in its plane about that point.



Let $AB=2a$, $AD=2b$, and 2ρ the density of the liquid. The point E is in the surface of the liquid, as the lamina floats with half its area immersed. Let H be the C. G. of BK, and H' of the triangle EKF. Draw GM, HN perpendicular to EF.

In the displaced position, the buoyancy of the lamina is equivalent to the weight of the liquid BK upwards through H, and that of EKF upwards through H'. Taking moments of the forces acting upon the lamina, about E, we have the equilibrium stable or unstable according as the moment of the buoyancy prevails over that of the weight of the lamina, or otherwise. Now as θ is small, we have,

$$EN=EG\cos\theta-HG\sin\theta=b-\frac{a}{2}\theta \text{ approximately,}$$

$$EKF=2b\cdot 2b\frac{\theta}{2}=2b^2\theta,\qquad EH'=\tfrac{2}{3}\cdot 2b=\frac{4b}{3}$$

$$\text{and } EM=EG\cos\theta=EG=b$$

Hence the moment of the buoyancy

$$=2g\rho\left\{2ab\left(b-\frac{a}{2}\theta\right)+2b^2\theta\cdot\tfrac{4}{3}b\right\}=2g\rho(2ab^2-a^2b\theta+\tfrac{8}{3}b^3\theta).$$

And the moment of the weight $=g\rho\cdot 4ab^2$

$\therefore$ The equilibrium is stable or unstable, according as

$$2ab^2-a^2b\theta+\tfrac{8}{3}b^3\theta > \text{ or } < 2ab^2$$

$$i.\ e. \text{ as } \tfrac{8}{3}b^3 > \text{ or } < a^2b$$

$$i.\ e. \text{ as } \frac{b^2}{a^2} > \text{ or } < \tfrac{3}{8}.$$

EXERCISES.

(1) If a cubic foot of water weigh 1000 oz., and a cube whose edge is 18 inches, weigh 2,250 ounces, how far will a cylinder whose length is three inches, and formed of the same material as the cube, sink in water? Ans. 2 in.

(2) A raft, 37 yards long by 18 yards broad and 16 inches deep, floats when submerged one third of its depth; with what weight must it be loaded before it sinks. Ans. 333,000 lb.

(3) A uniform body floats freely in a fluid whose density is twice as great as its own; prove that it will float in equilibrium, if its position be inverted.

(4) A weight suspended by a string from a fixed point is partially immersed in water; will the tension of the string be increased or diminished as the barometer rises? State your reasons.

(5) What volume of cork ($sp.gr.=\cdot 25$) must be attached to 10lbs. of iron ($sp{\cdot}gr=7\cdot 5$), in order that the whole may just float in water. Ans. 319·49 cb. in.

(6) A solid displaces $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ of its volume respectively when it floats in three different fluids. Find the volume it displaces, when it floats in a mixture formed of equal volumes of the fluids. Ans. $\frac{1}{3}$.

(7) The height of a cylindrical vessel is 12 inches, and the area of its base 20 inches; if the vessel be half filled with water, prove that a solid cylinder of specific gravity ·8, the area of

whose base is 15 inches and whose height is 10 inches, will just float in the water without causing any of it to flow out of the vessel.

How much water will flow out if the height of the solid cylinder were $12\frac{1}{2}$ inches ? Ans. 30 cb. in.

(8) A body in the form of an equilateral cone and hemisphere on the same base floats in a liquid of double the density of the body; find the position and nature of equilibrium.

Ans. Hemisphere immersed; neutral.

(9) A cylinder 1 ft. long floats $\frac{2}{3}$ immersed in a trough of water. How much of it will be immersed in water when a liquid of sp. gr. ·2 is poured upon the surface of the water just sufficient to cover the cylinder? Ans. 7 in.

(10) Upon a floating hemisphere is affixed another hemisphere of equal size, but (1) of less, and (2) of equal, and (3) of greater density. Describe the equilibrium of the sphere in each case.

(11) A tumbler is nearly filled with tepid water; a lump of ice is then placed in the water; the moment the ice floats freely, the water is exactly level with the rim of the tumbler. Explain whether the water will or will not overflow as the ice melts.

(12) Two bodies, whose specific gravities are 6 and 7 respectively, balance when suspended at the extremities of a weightless rod 29 inches long supported on a moveable fulcrum placed at its centre. In what direction and to what distance must the fulcrum be moved so that the bodies may balance each other when both are totally immersed in a fluid of *sp. gr.* 2.

Ans. $\frac{1}{2}$ in. towards the second body.

(13) A cube whose density is twice that of water, is suspended by a string attached to the centre of one of its faces, and hangs so that this point is in the surface of water at rest. Compare the tension of the string with the whole pressures on all the immersed faces of the cube, and with the resultant horizontal pressures on them in the direction of a diagonal of the upper face. Ans. $2:1$; $\sqrt{2}:1$.

(14) A fluid, whose specific gravity is 2, rests upon another whose specific gravity is 4. A cylinder whose sp. gr. is 3 is wholly immersed with its axis vertical. Find the position of equilibrium. Ans. The cylinder is bisected by the common surface.

(15) A uniform rod of length $(a+b)$, capable of turning about the end which is not immersed, rests inclined to the

horizon with the portion b immersed in water; show that its specific gravity is $\frac{b(2a+b)}{(a+b)^2}$.

(16) An indefinitely thin hollow sphere A, and a solid sphere B, which are connected together, float totally immersed in a liquid. If A were solid and B hollow, the combination would float in the same liquid with A totally and B half immersed; show that the radii of the spheres are in the ratio $1 : 2^{\frac{1}{6}}$.

(17) A body formed of a cylinder and a hemisphere upon the same base floats in water in neutral equilibrium with one-half of the redius of the hemisphere immersed; find its density.

Ans. ·15.

(18) A spherical shell, of which the external diameter is 18 inches, just floats in a fluid; what is the internal diameter? (sp. gr. of the fluid = ·91, and that of the material of the shell = 2·16).

Ans. 15 in.

(19) A cylinder which floats in water under an exhausted receiver has $\frac{3}{4}$ of its axis immersed. Find the alteration in the depth of immersion, when air whose sp. gr. is ·00125 is admitted.

Ans. About $\frac{1}{10}$ of the axis is further depressed.

(20) A triangular lamina ABC floats in a liquid, with its plane vertical; if the surface of the liquid meets AB, AC in D and E, show that the line joining the middle points of BC and DE must be vertical.

(21) How much of its weight will 1 cwt. of cast iron lose, if immersed in water? (sp. gr. of cast iron being 7·25).

Ans. 15·44 lb nearly.

(22) A uniform rod attached by one end to a vertical string, rests with $\frac{2}{3}$ of its length immersed in a fluid. Shew that the density of the rod : density of fluid, as 8 : 9; and that the tension of the string is one fourth of the weight of the rod.

(23) A cylinder of uniform density floats vertically in a fluid with $\frac{2}{3}$ of its axis immersed. The sp. gr. of the fluid being ·75, find that of the cylinder; also find the weight which must be placed on the cylinder in order just to submerge it.

Ans. ·5; half the weight of the cylinder.

(24) A raft 30 feet long, 10 feet wide and 20 inches deep, is made of material whose specific gravity is ·35; find the greatest weight it can support in water. Ans. 20812·5 lb.

(25) A thin rod of weight W and sp. gr. s, floats vertically in water. Find the minimum weight of a heavy particle to be

attached to the lower end so that it may float in stable equilibrium. Ans. $W\left(\frac{1}{\sqrt{s}}-1\right)$.

(26) A cylindrical pontoon 40 feet long and 5 feet in diameter is immersed to half its depth by its own weight. Find the additional weight it will bear before it is wholly immersed.
Ans. Weight of pontoon.

(27) A cone (semi-vertical angle 45°) floats vertex downwards in a fluid with $\frac{5}{6}$ of its volume immersed. A sphere of the same material, and radius equal to that of the base of the cone, is then attached to its vertex. Shew that $\frac{1}{\sqrt[3]{6}}$ of the axis is now immersed.

(28) A ship sailing from the sea into a river sinks two inches, but after discharging 40 tons of her cargo rises an inch and a half; determine the weight of the ship and the cargo together, the horizontal section for two inches above the sea being invariable; (sp. gr. of sea-water $=1\cdot025$).
Ans. $2186\frac{2}{3}$ tons.

(29) A conical vessel floats in water with its vertex downwards and a certain depth of its axis immersed; when filled up to the depth originally immersed, it sinks till its mouth is on a level with the surface of the water. Find what portion of the axis was originally immersed. Ans. $\frac{1}{\sqrt[3]{2}}$.

(30) A body floats half immersed in a fluid, but is $\frac{3}{4}$ths immersed when floating in a mixture of equal volumes of that fluid and water. Find its density. Ans. $1\cdot5$.

(31) A cubic inch of one of two liquids weighs m grains and the other n grains. A body immersed in the first liquid weighs p grains; and immersed in the second weighs q grains. What are its true weight and volume. Ans. $\frac{np-mq}{n-m}$; $\frac{p-q}{n-m}$.

(32) A triangular lamina ABC, right angled at C, is attached to a string at A, and rests with the side AC vertical, and half its length immersed in a fluid. Shew that the density of the fluid : the density of the lamina $=8:7$.

(33) A body floating on an inelastic fluid is observed to have volumes V_1, V_2, respectively above the surface at times when the density of the surrounding air is ρ_1, ρ_2: find the density of the fluid. Ans. $\frac{V_1\rho_1-V_2\rho_2}{V_1-V_2}$.

CHAPTER VII.

On Air and Gases.

80. Air and gases are called elastic fluids, from the property which they possess of continually tending to occupy a greater volume. They differ from liquids in as much as they are readily compressible when pressed, and expand unlimitedly when the pressure is withdrawn.

It is now well known that solids, liquids, gases are but different forms of matter. By the application of heat, which overcomes the cohesion between the particles, solids gradually assume the liquid state, and liquids become gases; whereas on the contrary, all gases by the application of cold and sufficient pressure assume the form of liquids, and even solids. The word gas however is confined to those bodies which remain in the aeriform state under *ordinary* temperature and pressure, while the aeriform state of the bodies which are *ordinarily* liquids, is called *vapour*. So long as the body is in the liquid state, it is subject to the laws and properties of liquids; and when in the gaseous state, it is subject to the peculiar laws of gases. The atmospheric air may be taken as the common type of gases.

The expansibility and compressibility of gases may be illustrated by simple contrivances. Take a common syringe with a tight fitting piston. When the rod of the piston is forced down in the tube, the air beneath is compressed, and a continually greater force will be needed to force the piston through any given length of the tube. As soon as the force is removed, the air regains its original volume, and the piston is forced back.

The expansibility of air is best illustrated by filling an elastic bladder with ordinary air, and placing it within the receiver of an air pump. As the air within the receiver is gradually withdrawn, the bladder distends from the elastic force of the air, no longer counterbalanced by the pressure of the surrounding air. When air is admitted into the receiver, the bladder resumes its original form.

81. Gases are, like liquids, heavy bodies, although on account of the smallness of the mass which ordinarily comes under our notice, its weight is not taken into account. That air has weight may be directly verified by comparing the weight of a hollow glass globe with that of the same globe exhausted of air by means of an air pump. By delicate measurement of this kind, it has been found that 100 cubic inches of dry air under the ordinary atmospheric pressure (indicated by 30 inches of mercury in the barometer and temperature 16°C) weigh about

31 grains ; or that water is about 771 times heavier than air, both being of the temperature 0°*C.* In the same manner the weights of other gases are determined. The intrinsic weights are thus found to be different for different gases.

82. We have stated that a gas always tends to expand in volume. Hence the particles of a gas confined in a vessel, exert pressure on each other, and on the sides of the vessel. If the gas be not confined in a vessel, the result will be an indefinite expansion of the gas. The pressure exerted at any point of a gas may therefore be regarded from two points of view :

(1) Suppose the gas is contained in a cylindrical vessel. The expansive force of the gas is checked by the reaction of the sides of the vessel. The greater the expansive force of the gas, the greater must be the reaction of the sides to keep it confined. Now the expansive force of a volume of gas depends upon its temperature. An increase of temperature tends to increase the volume, and the sides of the vessel have therefore to sustain greater pressure. The reaction of the sides is transmitted through the gas, and an increase of pressure at every point is the consequence. Take for instance a layer of gas in the cylinder ; it is pressed on all sides not only by the weight of the gas above it, but also by the reaction of the sides transmitted through the gas. The pressure at different points will, under this view, be accordingly different, and depend upon the weight of the gas as well as its volume and temperature. If however the weight of the gas be neglected as being small in comparison with the expansive force, we have the case of a fluid not acted on by any impressed force, and the pressure at every point will be the same throughout, whether within the mass, or or at the sides of the vessel. (*See Art.* 10). In all cases where the quantity of the gas is not considerable, we are justified in considering the pressure constant at every point in its volume. But for large quantities, the pressure due to the weight is considerable and must be allowed for.

(2) Take the case of a gas free to expand, as the atmospheric air. We will best comprehend the pressure at any point of such a gas by conceiving it to be contained in a vertical cylindrical vessel of indefinite length, open at the top, and closed by a moveable piston without weight. It is evident the gas will increase in volume so long as the expansive force at any point is just counter-balanced by the weight of the column pressing it from above. Thus consider a thin layer somewhere in the cylinder. This layer sustains the weight of the column of gas over it, and transmits that weight to the gas beneath it, as well as to the curved surface of the cylinder which contains it. Thus the

pressure at any point of the gas obeys the same laws as those which hold good for liquids at rest, and is entirely due to the weight of the vertical column of the gas above that point. The pressure accordingly increases from the top of the column to the base. Also the effect of the weight of any column will be to compress the gas beneath, and therefore the density of any layer will also increase from the top to the base.

83. We are now in a position to account for the atmospheric pressure. The earth is surrounded on all sides by layers of air, which partakes of the rotatory motion of our globe, and would remain relatively at rest, but for local circumstances which produce winds. We will neglect these local perturbations, and whenever we treat of atmosphere, it must be supposed to be at rest.

The height of the atmosphere must be limited; for the expansive force of air decreases in proportion as it expands, and at a certain height where the expansive force is just counteracted by gravity, further vertical expansion cannot take place. Upon any horizontal area therefore we will suppose a cylindrical column of air extending to the free surface, the density decreasing from the bottom to the top, and the pressure on the area is the weight of the column above it. For small heights however, the difference of densities being very small, we may regard the atmosphere as a sea of uniform fluid, to which all the laws of hydrostatics established for liquids in the previous chapters are applicable.

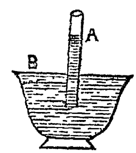
84. Although the density of the air is very small in comparison with that of water, the whole atmosphere exercises a considerable pressure on the surface of the earth, and on bodies lying upon it. The existence of such pressure is well illustrated by what are called *Magdeburg* hemispheres. This apparatus consists of two hollow brass hemispheres, the edges of which are made to fit each other tightly, and are well greased; one of the hemispheres is provided with a stop cock which can be screwed to an air pump; on the other there is a handle.

As long as the hemispheres contain air, they are separated easily, for the atmospheric pressure on the exterior surface is counterbalanced by the pressure of the air in the interior. But when the air from the interior is exhausted, the hemispheres cannot be separated without a powerful effort. For instance, if the diameter of the sphere be 4 inches, the force necessary to separate them is found to be more than 180 lbs. In fact the force of separation will be equal to the resultant atmospheric pressure on a circle of the same radius as the sphere. Also as the same force is always necessary to separate them in whatever

position they are held, the experiment furnishes another illustration of the fact that the pressure of the atmosphere at a point is equal in all directions.

85. The quantitative action of the atmosphere was distinctly ascertained by the experiment of Torricelli, first undertaken in 1643.

A glass tube of about a yard long, open at one end and closed at the other, is filled with mercury; the open end being closed by the thumb, the tube is inverted, the open end immersed in a cup of mercury, and the thumb is removed. The tube being in a vertical position, the column of mercury sinks, leaving a vacuum at the top, and after oscillating for some time, finally rests with its surface at A at a height of about 30 inches above the surface of mercury in the cup, if the experiment be carried out at some place not much above the level of the sea. It is clear that the column of mercury in the tube is supported by the atmospheric pressure acting on the surface of the mercury in the cup, and transmitted through the mercury.



The upper part of the tube is a vacuum, and hence there is no contrary pressure of the air upon the column AB. The weight of the column AB of mercury therefore provides us with the means of measuring the atmospheric pressure on the sectional area of the tube. And as we have seen that the pressure of the atmosphere on a given area is equal to the weight of the column of air above that area extending in height to the free surface of the atmosphere, it follows that the weight of a column of air is equal to that of mercury on the same base of the height AB.

That the column AB is supported by the atmospheric pressure on the surface of the cup is verified in various ways. Let the top of the tube be opened; the column AB being now pressed equally on the top and bottom by the atmosphere, the weight is no longer counterbalanced, and the column immediately sinks to the level of the mercury in the cup. Again, if the experiment be made at different heights above the level of the sea, as was done by Pascal in 1646, it is found that the column is less in proportion as the height is greater, as might have been expected, since the pressure of the atmosphere is less as we ascend. Also if experiments be made with liquids other than mercury (making the tube proportionately long), the column of the liquid thus supported upon the same base, is always found to have the same weight, a fact which at once points to the upward atmospheric pressure on the column as the force which supports it in every case. The column of water thus supported

is found to be about 33 feet in length, or about 13 times the column of mercury.

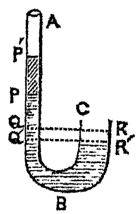
Instead of ascending to different heights in order to vary the atmospheric pressure on the surface of the cup, the same result may be obtained by placing the tube and the cup inside the receiver of an air-pump. As the exhaustion of the receiver proceeds, the pressure on the surface of the cup diminishes, and the column of mercury is found to descend continually. If a complete vacuum is produced in the receiver, the column sinks to the level of mercury in the cup.

86. The experiment of Torricelli led immediately to the construction of the Barometer for measuring the pressure of the atmosphere. Various forms of this instrument have been proposed from time to time, but the following form known as the siphon barometer is commonly used in practice.

ABC is a bent tube the branches of which are straight. The portion *BC* is of much greater diameter than *AB*, and the height of the tube is about 32 or 33 inches. The end *A* is closed, and the end *C* is open. The branch *AB* contains mercury which extends to the branch *BC*. The upper part of the branch *AB* is vacuum.

Let the surface of mercury in *AB* be *P*, and let the horizontal plane through *R* the surface in *BC*, meet *AB* in *Q*. The column of mercury *PQ* is supported by the atmospheric pressure on the surface *R*, transmitted through the mercury, the pressure at a point in *R* being equal to the pressure at a point in *Q* in the same horizontal plane. Hence if a be the sectional area of *AB*, σ the specific gravity of mercury, and π the atmospheric pressure on a unit of area, then

$$\pi a = \sigma \,.\, PQ.\, a$$
$$\therefore\ \pi = \sigma \,.\, PQ.$$



The length *PQ* is measured by means of a graduated scale sliding along the length of the tube *AB*. Sometimes the tube itself is graduated, but then in order to measure the column *PQ* accurately, the length of each graduation must be modified as follows.

Suppose the surfaces of mercury in the two branches rest at *P* and *R* at one moment, so that the column *PQ* measures the atmospheric pressure.

Let the surface rise to *P′* in consequence of an increase of atmospheric pressure; it is clear that the surface *R* will fall to *R′*, and column *P′Q′* will now measure the atmospheric pressure.

The increase in the length of the column of mercury due to the increase of the pressure is $PP' + QQ' = PP' + RR'$.

Let K and k be the sectional areas of the shorter and longer branches of the tube. Then since the quantity of mercury is unchanged,

$$PP'.\,k = RR'.\,K$$

$$\therefore \text{ increase in the column} = PP'\left(1 + \frac{k}{K}\right)$$

Now the apparent rise of mercury is PP', whereas the real rise is $PP'\left(1 + \frac{k}{K}\right)$. Hence the real height of mercury is greater than the apparent height in the ratio of $1 + \frac{k}{K} : 1$.

For convenience of reading, the graduations are so made that the distance actually measured above the zero point is marked larger in the ratio of $1 + \frac{k}{K} : 1$, so that the reading at once gives the real height above zero mark.

87. In order to determine by means of the Barometer the atmospheric pressure at any moment with any degree of scientific accuracy, several practical difficulties occur with regard to the construction, adjustment and reading of the instrument, and render certain precautions, and corrections necessary. We can here but enumerate some of them, and refer the student for fuller information to special works on the subject.

(*a*) The branches of the tube should be perfectly straight and of uniform bore. The bore must not be very narrow, otherwise capillary attraction will interfere with the proper reading of the instrument.

(*b*) The tube should be perfectly vertical; if it is inclined, the column of mercury will be elongated, and the reading on the scale will be too great. The perfect verticality is secured by a contrivance called *Cardan's suspension.*

(*c*) The upper part of the longer branch of the tube should be quite void of air or aqueous vapour; otherwise its elasticity will depress the column of mercury. For if p be the pressure of the air or vapour left in AP, the atmospheric pressure $\pi = p + \sigma PQ$. This requisite is in a great measure secured by pouring in mercury little by little, and boiling it each time until the tube is quite full.

The portion AP can never be a perfect vacuum. It is at best filled with the vapour of mercury, and in this state it is called the Torricellian vacuum.

(*d*) A difficulty arises in reading the column PQ from the circumstance that mercury having no adhesion with the glass tube, the surface at P is not plane, but of the form of a curved meniscus with convexity upwards. The dimension of the meniscus above P depends upon the sectional area of the tube, so that for a known sectional area, the volume of the meniscus may be known by reference to tables formed for the purpose, and the weight of the column above Q is determined.

(*e*) In all barometer readings, a correction must be applied for temperature. Mercury, like most other subtances, expands with a rise of temperature and contracts with a fall of temperature. The specific gravity of mercury is therefore different at different temperatures, and the same atmospheric pressure at different temperatures will be indicated by different columns in the barometer. Accordingly in each observation, the height observed should be reduced to a standard temperature. The choice of this temperature is quite arbitrary, but that at $0^\circ C$, or the temperature of melting ice, under the standard atmospheric pressure is usually taken as the standard. The reduction is thus effected :

Let σ_0 and σ_t be the specific gravities of mercury at $0^\circ C$ and $t^\circ C$ respectively. Now from the definition of temperature derived from the mercurial thermometer, that is, assuming that the expansion of mercury is proportional to the increase of temperature, it is found by experiment that the expansion of mercury for an increase of $1^\circ C$, the expansion is $\frac{1}{5550}$ or ·00018018 of its volume at $0^\circ C$. Denote this small fraction by a. Then if V_t be the volume of mercury at $t^\circ C$, we have

$$V_t = V_0 + V_0 at.$$

$$\therefore \quad \frac{V_t}{V_0} = 1 + at.$$

$$\text{Also } \frac{\sigma_t}{\sigma_0} = \frac{V_0}{V_t} = \frac{1}{1+at} = (1+at)^{-1} = 1 - at \text{ nearly.}$$

$$\therefore \quad \sigma_t = \sigma_0(1 - at).$$

Hence if $PQ = h$ be the observed height of the barometer at $t^\circ C$,

$$\pi = \sigma_t h = \sigma_0(1 - at)h.$$

88. The average height of the barometer at the level of the sea for temperature of $16°C$, is 29·9 inches nearly. This is usually called the *standard* atmospheric pressure, through which the readings at different heights, and at differents times, are compared with one another. This height is however subject to continuous daily, monthly and annual variations. Changes of weather and latitude also produce change in the height of the column, but these are matters of meteorology with which we have at present no concern. The height of the barometer gives the atmospheric pressure at the time of observation. Thus if π be the atmospheric pressure on a square inch, and the height of the barometer be 30 inches, $\pi = 13·568 \times 30 \times \frac{1000}{1728}$ oz. $= 14·7$ lbs. nearly.

89. It is clear that any liquid may be used in constructing a barometer, but the great density of mercury renders it the most convenient for the purpose. Being the densest of all known liquids, it stands at the least height for a given pressure. We can easily determine the height of any other liquid if used in a barometer.

Let σ and σ' be the densities of mercury and the other liquid, h and x the heights of the barometers for a given pressure,

$$\text{then } \sigma h = \sigma' x \qquad \therefore\ x = \frac{\sigma}{\sigma'} h$$

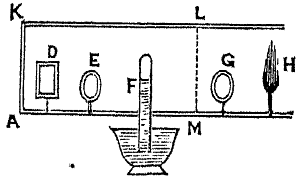
Thus for the water barometer, $\frac{\sigma}{\sigma'} = 13·568$, and taking $h = 30$ inches, we have

$$x = 13·568 \times 30 \text{ inches} = 34 \text{ feet nearly.}$$

Glycerine Barometer. Recently barometers have been constructed with *glycerine*, which seems well adapted for the purpose; for its vapour has a very low tension at ordinary temperatures, and its freezing point is much below that of mercury. The mean coefficient of expansion by heat is ·000303 for $1°F$. At sea level the pressure of the atmosphere supports a column of glycerine of nearly 27 feet in height; accordingly the tube of the barometer is made about 29 feet, the upper 4 or 5 feet of which is glass tube of one inch diameter, the rest being metallic.

The quantity of glycerine required for such an instrument is about a gallon. The glycerine barometer is a simply constructed and easily managed instrument. As it is comparatively a new instrument, its value as a piece of scientific apparatus for precision is however not fully determined.

90. *Self registering barometer.* In several meteorological observatories, the rise and fall of mercury in the barometer is automatically registered night and day on strips of paper sensitized by a photographic process. The registered papers are called Barograms. A rough description of this is given below.



AM is a straight slab supported upon piers. Sensitized paper is wrapped round the cylinder D, which is turned round by means of clock-work. A light from H, focussed by a lens G, passes over the column of mercury at F in the barometer. This light is again focussed by the lens E, before it falls on the sensitized cylinder. KLM is a covering to prevent scattered light from discoloring the paper on the cylinder. As the light is cut off by the rise of mercury, and is allowed to pass by its fall, the height of mercury in the barometer at any moment will evidently be indicated by the curve bounding the portion of the paper acted on by the light. By means of a suitable graduated scale, the curve traced out on the paper may be reduced to figures.

The *Standard Aneriod Barometer* is constructed in the following manner :—A nearly flat metal box is exhausted of air, and its upper surface or lid is firmly held by a powerful spring, which is connected with the moveable hands of a graduated dial by means of levers and chain. The small motion of the lid produced by the varying pressure of the air is multiplied by the levers, thus giving an extended reading on the margin of the dial outside. At the back of the instrument a screw head is visible, which if slowly and carefully turned, will enable the aneroid to be adjusted to a mercurial standard barometer.

91. *Temperature.* Heat manifests itself by the sensation it produces in our mind, but such sensation is obviously not adapted for quantitative analysis. Temperature is measured by the expansion which heat produces in the volumes of bodies heated. It is a general rule that bodies expand under the action of heat and contract when that heat is taken away. But it must not be supposed that all bodies serve equally well as thermometers on this account. For the measurement of all ordinary temperatures mercury is the best substance for the construction of a thermometer. Its advantage over any other substance consists in the following properties :—

(*a*) It is a liquid, and as a general rule liquids are better suited for thermometers ; for the expansion of a solid is too small, and that of a gas too great.

(*b*). Mercury boils at a sufficiently high temperature, and freezes at a sufficiently low temperature; so that a long range of temperatures, sufficient for all ordinary purposes, can be indicated by a mercurial thermometer.

(*c*). The expansion of mercury even for a small increment of temperature is sensible, so that a mercurial thermometer is capable of detecting a small difference of temperature.

(*d*) Mercury rapidly adapts itself to the temperature of the body with which it is in contact.

For the measurement of low temperatures, where mercury freezes, alcohol is used. For high temperatures, where mercury is vaporized, the expansion of a bar of metal is used. Thermometers of the latter kind are called pyrometers.

The mercurial thermometer is formed of a thin glass tube of uniform bore, terminating in a bulb. The upper end is at first open, and mercury is poured in through it, little by little, the air inside being driven out by repeated boiling over a spirit lamp. When the bulb and tube are filled to overflowing at the boiling temperature of mercury, the top is hermetically sealed, and the instrument allowed to cool; the mercury then contracts, leaving a vacuum at the top. Experience shows that it is not fit for graduation till some three or four years after filling. Experiment shows that water under a given atmospheric pressure boils, *i. e.*, gives off steam, at a constant temperature, and also freezes at a constant temperature. Accordingly the tube is dipped in the vapour of water boiling at the standard atmospheric pressure, and the top of the mercury in the tube is marked as the *boiling* point; similarly the *freezing* point is marked where the mercury rests when dipped in melting ice. The length of the tube between the freezing and the boiling point is now divided into a number of equal parts, each called a degree. There are many modes in which this is done. Thus in the *centigrade* thermometer, the freezing point is marked zero, and the boiling point 100, and the intermediate portion is divided into one hundred equal portions or *degrees*. The stem above 100 and below zero is also graduated in like equal divisions, the degrees below zero being expressed by the negative sign.

In Fahrenheit's thermometer, the freezing point is marked 32°, and the boiling point 212°, and the intermediate portion is divided into 180 equal parts or degrees. Graduation on the same scale is extended below and above that portion. The zero mark in this thermometer corresponds to the temperature of a freezing mixture consisting of equal weights of ice and salammoniac.

In Reaumur's thermometer, the freezing point is zero, and the boiling point is 80°. This thermometer is not much used at present.

Besides these, there is a thermometer called DeLisle's, in which the boiling point is marked zero, and the freezing point 150. This is much used in Russian scientific investigations.

A particular temperature being indicated by one of these thermometers, the reading can be readily converted into those in the other scales. Thus

Let C and F denote the readings in the centigrade and Fahrenheit's thermometers corresponding to a given temperature. Then it is evident that $\frac{C}{100}=\frac{F-32}{180}$, or $\frac{C}{5}=\frac{F-32}{9}$.

Example. What temperature is indicated by the same reading in either thermometer?

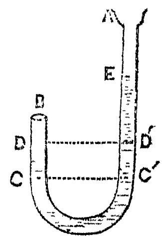
$$\text{Here } C=F$$

$$\therefore\ 9C=5C-160 \quad \therefore\ C=-40°.$$

Obs. It should be noticed that the expansion of mercury is assumed to be proportional to the increase of temperature as measured by the mercurial thermometer. Mercury freezes at $-39{\cdot}4°C$, and boils at about 350°; hence a mercurial thermometer cannot be used for ascertaining temperatures beyond these limits.

92. The relation between the pressure and volume of a gas was first investigated independently by Boyle and Mariotte, and the result of their experiments established the following law, known as Boyle and Mariotte's Law.

"*The temperature remaining the same, the pressure at a point of a given quantity of gas varies inversely as the volume it occupies.*"



(1) To verify the law for the compression of the gas, take a bent glass tube, fixed to a vertical support. The legs of the tube are straight, and at first open. A small quantity of mercury is poured into the tube, which will evidently rest with the surfaces C and C' in the same horizontal plane. The end B is now closed, and, more mercury poured in through the end A. The effect of this is to compress the air in BC, the surfaces of the mercury resting at D and E in the tube. Let the height of the barometer observed at the time of the experiment be h. Draw the horizontal plane DD'.

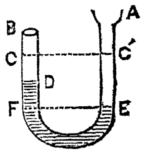
Now if BC be divided by a scale into portions of equal capacities, and the height DE of the mercury be observed, it will be always found that if $BD = \frac{1}{2}BC$, $D'E=h$; if $BD=\frac{1}{3}BC$, $D'E=2h$; if $BD=\frac{1}{4}BC$, $D'E=3h$, and so on, if $BD=\frac{1}{n}BC$, $DE=(n-1)h$.

Before the end B was closed, the pressure of the air in BC was σh. But when the air in BC is compressed into BD, so that $BD=\frac{1}{n}BC$, the pressure at any point in BD = pressure at D = pressure at $E+\sigma D'E=\sigma(h+D'E)=\sigma nh$.

$$\therefore \frac{\text{The pressure at a point in } BD}{\text{The pressure at a point in } BC}=\frac{\sigma nh}{\sigma h}=\frac{n}{1}=\frac{BC}{BD}.$$

This verifies the law for the compression of the air.

(2) To verify Boyle and Mariotte's law for the dilatation of gas, take a bent tube of which both branches are long; pour in mercury, so that the surfaces rest at C, C', in the same horizontal plane. Note the length BC, and after closing B, withdraw some mercury through the branch A, and let D and E be the surfaces of mercury.



Draw the horizontal plane EF, and measure the height DF. It will be always found that if $BD=nBC$, $DF=\left(1-\frac{1}{n}\right)h$ or $h-DF=\frac{h}{n}$.

But the pressure at a point in BD = pressure at $F-\sigma.DF$ $=\sigma(h-DF)=\frac{\sigma\cdot h}{n}$.

$$\therefore \frac{\text{pressure at a point in } BD}{\text{pressure at a point in } BC}=\frac{1}{n}=\frac{BC}{BD}$$

which verifies the law for dilatation of the air.

Obs. In the same manner the law may be verified for gases other than atmospheric air. The temperature of the gas must be carefully kept constant throughout the experiment. This is theoretically difficult to attain, since the compression of gas developes heat, and the dilatation occasions a reduction of tem-

perature. But for all practical purposes, where the change of temperature is small, the law may be applied without much error. Experiments have also shewn that the law is true only when the pressure produced by compression does not exceed a certain very high limit. Above that limit, the law does not strictly hold. Again, for all gases that do not readily liquify under pressure, as hydrogen for example, the law is found to be a very close approximation to the results of experiments; but for easily liquifiable gases, such as ammonia, the volume is found to decrease more rapidly than the pressure increases.

93. Let p be the pressure at a point of a given quantity of gas, when its volume is V, and p' the pressure of the same when its volume is V'; ρ and ρ' the densities of the gas when the volumes are V and V' respectively.

Then by Boyle's law

$$\frac{p}{p'} = \frac{V'}{V}.$$

But the mass of the air being the same, $V\rho = V'\rho'$.

$$\therefore\ p = \frac{p'}{\rho'}.\rho = k\rho \qquad \text{where } \frac{p'}{\rho'} = k.$$

The quantity k is constant for the same gas, and is determined by the simultaneous observation of density and pressure. It is usual to take p' as the pressure indicated by a barometric height of 29·92 inches, and is called the standard pressure. The value of ρ' is observed under that pressure, and the value of $\frac{p'}{\rho'}$ or k is tabulated for different gases.

The relation $p = k\rho$ is expressed by saying that the pressure of a gas varies directly as its density, the temperature remaining unchanged.

Example. A weightless piston fits into a vertical cylinder of length a containing atmospheric air, and is initially at the top of the cylinder. Water being gently poured on the top, find how much water can be poured in before it will run over.

Let π be the atmospheric pressure, ρ the density of water, and x the depth to which the piston sinks, when the water runs over.

The pressure of the compressed air inside is then $\pi + g\rho x$

$$\therefore \text{ By Boyle's law } \frac{\pi + g\rho x}{\pi} = \frac{a}{a - x}.$$

If h = height of the water barometer, $\pi = g\rho h$

$$\therefore \frac{h+x}{h} = \frac{a}{a-x}$$

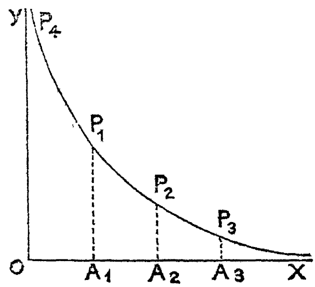
$$\therefore ah + ax - hx - x^2 = ah$$

$$\therefore x(a-h-x) = 0$$

$$\therefore x = 0 \text{ or } x = a - h.$$

This shews that a must be greater than h in order that any water can be poured in.

94. Boyle's law may be illustrated graphically by a curve, in the following manner.



Let V be the volume of a gas at a given temperature, and p the pressure at a point in it.

Then Vp = constant, so that when V is increased, p is proportionally diminished and *vice versâ*, the temperature being kept constant throughout.

Take OX and OY two straight lines at right-angles to each other. Along OX measure off OA_1 representing the volume V, and let OA_2, OA_3, &c. represent the gradual increased values of V. Through the points A_1 A_2 A_3 &c. draw ordinates A_1P_1, A_2P_2, A_3P_3 &c. representing the corresponding values of p. The curve formed by joining P_1 P_2 P_3 &c. will represent the relation between the volume and pressure of the gas at any instant.

By diminishing V, and noting the increased values of p, the portion of the curve P_1 P_4 may be described.

Students of conic sections will immediately recognise this curve as a *rectangular hyperbola*, whose asymptotes are OX and OY. The curve continually approaches on either side nearer and nearer to the asymptotes, but never touches them; this is representative of the fact, that no amount of expansion of volume will ever reduce the pressure to nothing, and that no increment of pressure will annihilate the volume. But it must be remembered that long before any such extreme case is

reached, the gas will very likely change its character, and Boyle's law will no longer be applicable.

95. It has been stated that the pressure of a gas depends not only upon the volume it occupies, but also upon its temperature. In order therefore to determine the pressure of a gas for a given volume and temperature, it is necessary to know the effect which a given temperature produces upon the elasticity of the gas.

The subject was investigated by Dalton and Gay Lussac, and their results were embodied in the following two laws ;

I. If the pressure be kept constant, the increase of temperature of every degree centigrade, produces in a given mass of air an expansion of ·003665 of its volume at $0°C$.

II. The decimal fraction ·003665 or ($\frac{1}{273}$ nearly), which is called the coefficient of expansion, remains nearly the same for all gases, and also for different pressures.

Expressed algebraically, the laws state, that if V_0 be the volume of a gas at $0°C$, and V_t its volume at $t°C$, then $V_t = V_0 + V_0 at = V_0(1 + at)$, where $a = ·003665$, provided the pressure of the gas throughout the expansion remain the same.

Recent experiments have shewn that these laws are not strictly accurate; but as deviations are found to be smaller and smaller as the gas is removed further and further from its point of liquification, it is assumed that the laws are strictly true for a perfect gas incapable of any liquification. The mathematical conception of such a *perfect gas* is of the same nature as that of a perfect fluid mentioned in the 1st chapter.

The above laws may be verified by taking a quantity of gas at 0_0C confined in a smooth cylinder by means of a freely moving air-tight piston of given weight. When the temperature of the gas is increased, its volume increases, forcing the piston out through a distance depending upon the increase of temperature.

96. The experimental law of Boyle, combined with that of Gay Lussac, enables us to express the relation between the simultaneous pressure, density and the temperature of a given quantity of gas. Thus

Let p, ρ and $t°C$ denote the simultaneous pressure, density and temperature of a given quantity of gas, and V its volume at that instant; and p_0 and ρ_0 the pressure and density of the same gas when the temperature was $0°C$, V_0 being its volume at that time.

We will suppose the variation of pressure and temperature to take place one at a time, the order being immaterial; thus we shall suppose that p_0 remaining the same, the volume V_0 increases to some volume V' in consequence of the rise of temperature from $0°$ to $t°$, and that from that moment the temperature being kept at $t°$, the volume V' changes to V in consequence of the change of pressure from p_0 to p.

Hence by Gay Lussac's law

$$V' = V_0(1+at)$$

and by Boyle's law

$$\frac{V}{V'} = \frac{p_0}{p} \quad \therefore \; p = \frac{V_0 p_0}{V}(1+at)$$

But the mass of the gas being the same, $\dfrac{V_0}{V} = \dfrac{\rho}{\rho_0}$

$$\therefore \; p = \frac{p_0}{\rho_0} \cdot \rho(1+at)$$

$$\text{But } p_0 = k\rho_0, \quad \therefore \; p = k\rho(1+at).$$

Obs. The relation $p = k\rho(1+at)$ may be established from the experimental laws directly by the rule of variation.

For by Gay Lussac's law

$V \propto 1 + at$ when p is constant.

But $V \propto \dfrac{1}{\rho}$, for the mass is constant.

$\therefore \; \rho \propto \dfrac{1}{1+at}$ when p is constant.

Again by Boyle's law, $\rho \propto p$ when t is constant,

$\therefore \; \rho \propto \dfrac{p}{1+at}$ when both p and t vary.

$\therefore \; p \propto \rho(1+at)$, or $p = k\rho(1+at)$, where k is a constant, to be determined by the observations of simultaneous values of p, ρ and t.

Thus if p_0 and ρ_0 be the pressure and density when $t=0$, we have from the above relation.

$$p_0 = k\rho_0.$$

The value of k is therefore the same as defined in Art. 93.

Ex. Let the volume of a gas at $t°C$ under the pressure p be V. Find the volume V' of the same gas at $t'°C$ under the pressure p'.

Let ρ and ρ' be the densities when the volumes are V and V' respectively, so that $\frac{\rho}{\rho'} = \frac{V'}{V}$.

Now $p = k\rho(1 + at)$ and $p' = k\rho'(1 + at')$

$$\therefore \quad \frac{p}{p'} = \frac{\rho}{\rho'} \frac{(1 + at)}{(1 + at')} \qquad \therefore \quad V' = \frac{Vp(1 + at')}{p'(1 + at)}.$$

97. If we imagine the temperature of a gas diminished until its pressure vanishes, without any change of volume, the temperature so obtained is called *The Absolute Zero of Temperature.*

Let τ be the reading of the centigrade thermometer for the Absolute Zero of temperature, then

$$p = 0 = k\rho(1 + a\tau)$$

$$\therefore \quad \tau = -\frac{1}{a} = -273° \text{ nearly.}$$

Def. The difference between the readings of the thermometer for a given temperature of a gas and the Zero of absolute temperature, is called the *absolute temperature of the gas.* If $t°$ represents its temperature, reckoned from the zero of the centigrade scale, its absolute temperature is $273° + t°$.

Let T be the absolute temperature of the gas whose temperature is $t°C$. Let p be its pressure and ρ its density,

$$\text{then } p = k\rho(1 + at)$$

$$\text{and } o = k\rho(1 + a\tau)$$

$$\therefore \ p = k\rho a(t - \tau) = k\rho aT.$$

This shows that if the density, and therefore the volume, of a gas is constant, the pressure depends only on its absolute temperature. Also if V be the volume, $Vp \propto T$. Accordingly Gay Lussac's law may be simply stated thus: *The pressure of a gas being constant, its volume varies as its absolute temperature.*

98. *Mixtures of gases.*

Gases have the property of readily mixing with one another. Thus if a communication be established between two or more closed vessels containing gases, they at once begin to mix, whatever be their densities; and unless chemical action take place, the mixture will be complete in a time depending upon the relative densities of the gas.

Experiments have led to the following two laws governing the mixture of gases. They are also true for vapours. They were first established by Barthollet.

I. The mixture of the gases is homogeneous, *i. e.* the proportion of the gases in every unit of volume throughout is the same, and the greater the difference of density of the gases mixed, the more rapidly would the mixture be formed.

II. If gases of the same pressure and temperature be mixed, so that no change of volume or temperature is produced, the pressure of the mixture is the same as that of the gases severally.

Thus if gases of volumes V and V', at the same pressure p and same temperature t, be mixed, the temperature being not changed by the mixture, p will also be the pressure of the mixture whose volume is $V+V'$.

99. The following proposition illustrates the experimental laws stated in the preceding article.

Two gases of volumes V and V' and at pressures p and p' respectively are mixed together in a vessel of volume u; the temperature of the gases and of the mixture being the same. To determine the pressure of the mixture.

Let w be the pressure of the mixture. To apply the experimental law, we must first reduce both the gases to the same pressure w before mixture; thus by Boyle's law,

The volume of the first gas at $w = \dfrac{Vp}{w}$

The volume of the second gas at $w = \dfrac{Vp'}{w}$

The mixture of these volumes being now effected, its pressure is w, and the volume of the mixture is u.

$$\therefore\ u = \frac{Vp}{w} + \frac{V'p'}{w}, \quad \therefore\ uw = Vp + V'p'.$$

If $V = V'$, and the mixture be also reduced to the volume V,

$$\text{then } Vw = Vp + Vp'$$

$$\text{or} \qquad w = p + p',$$

in other words, the pressure of the mixture is then equal to the sum of the pressures of the gases.

The proposition may be extended to any number of gases. Thus if p_1, p_2, p_3 &c. be the pressures of a number of gases of the *same* volume and temperature, the pressure of the mixture $=p_1+p_2+p_3+$ &c. This principle is known as Dalton's law of mixture, and may be thus stated: "When a mixture of several gases, at the same temperature, is contained in a vessel, each produces the same pressure as if the others were not present."

Example. Two volumes V and V' of different gases at pressures p and p', and of temperatuae t°, are mixed together; the volume of the mixtuie is u, and its temperature is t'°, to determine its pressure.

If the temperature of the mixture had remained t°, its pressure would have been $=\dfrac{Vp+V'p'}{u}$.

Let the temperature of the mixture be now changed from t° to t'°, the volume remaining u, and let the pressure be w. Then by Art. 96.

$$\frac{w}{\text{pressure at } t^\circ}=\frac{1+at'}{1+at}$$

$$\therefore \quad w=\frac{(Vp+V'p)(1+at')}{u(1+at)}.$$

100. The behaviour of gases, and the various phenomena in connection with them, have been explained by modern mathematicians on the principles of dynamics. A description of what is called the *dynamic* or *kinetic theory of gases* is beyond the range of the present work. Roughly stated, a gaseous body is supposed to be a collection of very small globular particles of perfect elasticity, quite independent of one another, and constantly moving about in all directions with very great velocity. They impinge on one another, like ivory balls on a billiard table, and their motion is changed by impact. When they come in contact with the sides of the vessel containing the gas, their momentum is resisted, and to this resistance of momentum is the pressure of the gas supposed to be due. The number of such particles occupying a given volume, say one cubic inch, is so enormous, that in spite of their large velocities, they move through but small space, and at every step, have to encounter a series of successive impacts with other particles. It follows then that the pressure of a gas on an element of area in contact with it depends jointly upon the number of particles that impinge on it in a given time, and their average velocity of impact. The greater the number of particles in a given unit of volume, the greater is the *density* of the gas. For if m be the mass

of each particle, and n the number of particles in a unit of volume, the quantity mn is the measure of the *density* of the substance. The average velocity of impact of the particles depends upon the temperature of the gas; for it is to temperature that the kinetic energy of the gas is due. Hence if the temperature, and therefore the average velocity of the particles, be kept constant, the pressure of the gas at a point varies as its density alone, a conclusion verified by the experimental law of Boyle. Similarly other conclusions are deduced from the primary mathematical conception of the gas referred to above.

101. *Approximate determination of heights by barometric observations.*

It has been stated before that the column of mercury in the barometer is supported by the atmospheric pressure; and as we ascend to greater heights, atmospheric pressure diminishes, and the height of the barometer falls. It is therefore possible to connect the vertical height between two places with the heights of the barometer observed in those places. The problem is however complicated by several variable circumstances, such as (1) the variation of density and temperature of the atmosphere, which diminish as we ascend; (2) the variation of the force of gravity in the inverse square of the distance from the earth's center; (3) the different quantities of aqueous vapor at different heights. For small heights however, we will neglect all the other variations except that of density. For more accurate investigation, taking all the variations into account, the student must wait till he has learned the Integral Calculus.

Let z be the vertical distance between two positions, the observed barometric heights at the lower and the higher positions being h and h' respectively. The temperature is supposed constant throughout the height.

Divide the thin vertical cylinder of air between the positions into n equal horizontal layers, and suppose each layer homogeneous, which will actually be the case when n is large enough.

Let p_r and ρ_r denote the pressure and density of the r^{th} layer beginning from the lowest, so that $p_r = k\rho_r$. The r^{th} layer is kept at rest by the pressures at its top and bottom, and its weight;

$$\therefore\ p_{r-1} - p_{r+1} = \frac{z}{n} g\rho_r = \frac{gz}{nk} p_r$$

$$\therefore\ p_{r-1} = \left(1 + \frac{gz}{nk}\right) p_r, \quad \text{for } p_r = p_{r+1} \text{ nearly.}$$

Let $1+\frac{gz}{nk}=m$.

$\therefore$ Putting $r=1$, $r=2$, &c. we have $p_0=mp_1$, $p_1=mp_2$, $p_2=mp_3$, &c. &c., $p_{n-1}=mp_n$. Multiplying and cancelling like terms, we have

$$p_0=m^n.p_n$$

But p_0 or $p_1=g\sigma h$, and $p_n=g\sigma h'$, where σ is the density of mercury.

$$\therefore \frac{p_0}{p_n}=\frac{h}{h'}=\left(1+\frac{gz}{kn}\right)^n$$

$$\therefore \log_e\frac{h}{h'}=n\log_e\left(1+\frac{gz}{kn}\right)=n\left\{\frac{gz}{kn}-\tfrac{1}{2}\left(\frac{gz}{kn}\right)^2+\ldots\ldots\ldots\right\}.$$

$$=\frac{gz}{k} \text{ very nearly, when } n \text{ is indefinitely increased.}$$

$$\therefore z=\frac{k}{g}\log\frac{h}{h'}.$$

Let z' be the height of homogeneous atmosphere of density ρ, at a constant temperature t^0C; then $p=g\rho z'$, and by Boyle's law, $p=k\rho$, $\therefore \frac{k}{g}=z'$.

The numerical value of z' may be thus conveniently calculated: The temperature t^0 may be taken as the mean of the several observations taken at different heights.

If ρ_0 be the density of the air at 0^0C under the pressure p, we have by Gay Lussac's law of expansion,

$$\rho_0=\rho(1+at), \quad \text{or } \rho=\frac{\rho_0}{1+at}$$

$$\therefore p=\frac{g\rho_0}{1+at}z'.$$

Now, let the pressure p be indicated by the barometric column l of mercury at 0^0C, so that $p=g\sigma_0 l$, where $\sigma_0=$ density of mercury at 0^0C.

$$\therefore \frac{g\rho_0 z'}{1+at}=g\sigma_0 l, \quad \therefore z'=\frac{\sigma_0}{\rho_0}(1+at)\,l.$$

$$\therefore z=\frac{\sigma_0}{\rho_0}(1+at)\,l.\log\frac{h}{h'}.$$

For a given value of l, (say 29·9 inches), the value of $\frac{\sigma_0}{\rho_0}$ may be found by experiment, and z' is found to be about 26,000 feet.

102. We have stated in Art. 10, that if the weight of the gas be inconsiderable, the pressure at any point within it will be the same throughout the gas. But when a given quantity of heavy gas of uniform temperature is confined in a vessel, it will arrange itself in horizontal layers of different densities. The pressures of such gas at different heights will be different. The pressure at a particular point being known, the method of Art 101 will give the pressure at any other point.

Example.. A given weight W of a heavy gas is confined in a smooth vertical cylinder by a piston of given weight. To find the height of the piston from the bottom.

Let z be the required height, and W' the height of the piston together with atmospheric pressure upon the piston.* The sectional area of the cylinder being considered unit of area, the pressure of the gas on the top of the cylinder $= W'$, and the pressure on its bottom $= W + W'$,

$$\therefore\ z = \frac{k}{g}\log\frac{W+W'}{W'} = \frac{k}{g}\log\left(1+\frac{W}{W'}\right)$$

$$= \frac{k}{g}\left(\frac{W}{W'} - \frac{1}{2}\left(\frac{W}{W'}\right)^2 + \ldots\ldots\right)$$

$$= \frac{k}{g}\,\frac{W}{W'}, \text{ nearly, when } \frac{W}{W'} \text{ is very small.}$$

If p be the pressure at a point where the density is ρ, then $p = k\rho$.

$$\therefore\ z = \frac{p}{g\rho}\cdot\frac{W}{W'}.$$

103. *Tensions of surfaces exposed to fluid pressure.*

Def. The force which tends to separate two particles connected together, is called their *tension.* For instance, if a weight of P lbs. be suspended from a bar of metal whose sectional area is a square inches, then P lbs. is said to be the *tension* which the bar sustains for the section a; and $\frac{a}{P}$ lbs. is the average tension for one square inch. Hence the tension of a substance *at a point in a given direction* is measured by the tension exerted over a unit of area (perpendicular to the given direction) containing the point.

When a vessel contains fluid, the pressure at any point on the side of the vessel produces tension at that point; and if the vessel be not strong enough, it will break at that point along a line at right angles to the force of tension. It becomes necessary therefore in practical hydrostatics to determine the tension at a point on the surface of a vessel containing liquid or gas under pressure, *e. g.* the boiler of a steam engine. The determination of the tension generally for any form of the vessel and in any direction whatever belongs to higher branches of mathematics. It is however usual in elementary hydrostatics to exemplify the method of treatment in two very simple cases:

(1) A thin cylindrical vessel contains fluid under a given pressure; to determine the circumferential and longitudinal tensions at any point on its surface.

(*a*) To determine the circumferential tension at the point, consider the cylinder divided into two halves by a plane passing through the axis and the point; and imagine that these two halves are sewn together along the line of section, so that the tension produced by the pressure is just sufficient to separate them. Take two transverse sections of the cylinder enclosing a ring of the surface of breadth l, containing the point.

Let r be the radius of the cylinder, p the pressure of the fluid at the point, t the tension produced at the point on a unit of area, and e the thickness of the cylinder.

Then considering the equilibrium of either half of the ring, it is evident that the two equal tensions at the given point and at the opposite end of the diameter through it, together balance the resultant fluid pressure on the half ring; hence

$$2\, t.\, e.\, l. = 2\, r.\, l.\, p.$$

$$\therefore t = \frac{p.r}{e}.$$

(*b*) To determine the longitudinal tension at the point, consider the cylinder divided into two portions through the point by a plane perpendicular to the axis, and imagine the two portions sewn together along the circle of section, so that the longitudinal tension is just sufficient to separate them.

The tension along the circumference of section, being uniform throughout, if t' be the tension at the given point, $2\pi r.t'e$ is the total tension along it. Considering the equilibrium of either portion of the cylinder, it is evident that this whole tension balances the resultant fluid pressure on the interior of that portion. This latter is equivalent to the pressure on the area of the circle of transverse section at the point;

$$\therefore\ 2\pi r.t'.e = \pi r^2.p.$$

$$\therefore\ t' = \frac{p.r}{2e}.$$

$$\therefore\ t = 2t'.$$

Example. A cylindrical boiler, $\frac{1}{2}$ inch thick and two feet diameter, contains steam at 150 lbs. to the square inch; find the tension or strain at any point at the surface.

$$\text{Here } t = \frac{150 \times 12}{\frac{1}{2}} = 3600 \text{ lbs.}$$

$$\text{and } t' = \frac{t}{2} = 1800 \text{ lbs.}$$

(2) To find the tangential tension at a point of a spherical vessel containing fluid under a constant pressure.

Suppose the sphere be cut into two halves through the point along a great circle perpendicular to the direction of tension, and imagine the two halves sewn together along the circle of section, so that the tension is just sufficient to separate them. As before, the total tension along the sewed circumference $= 2\pi r.t\,e$. Considering the equilibrium of either hemisphere, it is clear that the total tension balances the resultant fluid pressure on its interior surface, which is equivalent to the pressure on its projection on the plane of section, *i. e.* on the circular area πr^2;

$$\therefore\ 2\pi r.t.e = \pi r^2 p.$$

$$\therefore\ t = \frac{p.r}{2e}.$$

Example. A spherical shell, 1 feet diameter and $\frac{1}{20}$ inch thick, is made of a substance which can sustain a strain of 1000 lbs on a square inch. Find the greatest fluid pressure which the shell can sustain.

Here, taking 1 inch as the unit of length,

$$t = 1000 \text{ lbs.}$$

$$\therefore\ p = \frac{2et}{r} = 2 \times \tfrac{1}{20} \times \tfrac{1000}{6} = 16\tfrac{2}{3} \text{ lbs.}$$

Obs. The pressure p inside the vessel producing tension, must be taken to be the excess of the internal over the external pressure.

Table of maximum or breaking tensile strength of certain common substances, in pounds, per 1 square inch of section :

Copper . .	60000	Cast Iron. .	16500
Wrought Iron	67200	Steel . . .	120000
Brass . . .	18000	Lead . . .	3300
Glass . . .	9400	Wood from	10000 to 18000.

EXERCISES.

(1) Air is confined in a vertical cylinder surmounted by a piston without weight, whose area is a square foot. What weight must now be placed on the piston that the volume of air may be reduced to half its dimensions, the temperature remaining unaltered ? Ans. 2100 lbs. nearly.

(2) If the pressure of a mass of gas be 30·275 inches of mercury, and its volume 100 cubic inches ; calculate its pressure, if under the same temperature it is allowed to expand to a volume of 387 cubic inches. Ans. 7·823 in.

(3) If the pressure of a gas contained in a given volume be represented by 32·5 inches of mercury at the temperature 58° Fahrenheit, what will it become if the gas be heated to 275° Fahrenheit ? Ans. 41·13 in.

(4) What degree of Reaumer's thermometer corresponds to 39° Fahrenheit ? What degree of Fahrenheit corresponds to 60° of a centigrade ? and what degree of a centigrade corresponds to 60° of Fahrenheit ?

(5) A barometer stands at 30 inches, and the sectional area of the tube is one square inch. A cubic inch of air is admitted through the mercury into the vacuum above, and the column is thereby depressed through 4 inches. Find the length of the vacuum. Ans. $3\frac{1}{2}$ in.

(6) The vacuum above mercury is 5 inches, and the sectional area of the tube is one square inch ; what volume of atmospheric air is passed up so as to depress the reading from 30 to 29 inches ? Ans. $\frac{1}{5}$ cub. in.

(7) To what depth must the top of a cylindrical diving bell, 10 ft. long be immersed that the water may occupy half the length of the cylinder, the height of the water barometer being 33 feet. Ans. 27 ft.

(8) If the readings of two equal barometers under the same circumstances at two observations be h, k and h' k'; compare the quantities of air left in their upper portions.

(9) A quantity of air in a cylinder is compressed into another cylinder, the radius and height of which are each one half of those of the other; the pressure sustained on the whole interior surface of the latter is twice as great as it was on the former.

(10) Fahrenheit's thermometer at Simla stands at the same height 55°, as a centigrade in the plains; determine the difference of temperature in Fahrenheit's degree. Ans. 76°.

(11) An oil barometer stands at $36\frac{1}{2}$ feet when the mercurial barometer stands at $29\frac{1}{2}$ inches. Find the specific gravity of oil, that of mercury being 13·5. Ans. ·91 nearly.

(12) If V and V' be the volumes of the same mass of gas, p and p' the pressures, and $t°F$ and $t'° F$ the temperatures, show

that $$V' = \frac{460+t'}{460+t} \times \frac{p}{p'} \cdot V.$$

(13) Two barometers have the same sections of their shorter tubes, each one square inch; and those of the longer tubes $\frac{1}{2}$ and $\frac{1}{4}$ of a square inch respectively. If in each the mercurry is at 29 inches above the zero of measurement at a pressure of $14\frac{1}{2}$ lbs. to the square inch, find the difference of height, above the zero, of the mercury in the two barometers, at pressure of $14\frac{3}{4}$ lbs. to the square inch, the zero being at the level of the mercury in the shorter tube at a pressure of $14\frac{1}{2}$ lbs.

Ans. $\frac{1}{30}$ in.

(14) For an increase in temperature corresponding to a rise of $1°C$, the expansion of mercury is $\frac{1}{5550}$ of its volume. If the barometer stands at 30 inches when the temperature is 0°, find to two places of decimals, the height of the barometer, in inches, for the same atmospheric pressure when the temperature is $30°C$.

Ans. 30·16.

(15) Two vessels cointain air at 2 and $\frac{1}{4}$ atmosphere pressures respectively. The capacities of the vessels are 3 and 18 cubic feet. What is the pressure in the vessel, when a communication is opened between them? Ans. $\frac{1}{2}$ Atmosphere.

(16) A horizontal cylinder containing air at the pressure of 15 lbs. on the square inch, is closed by a piston which rests at a distance of 10 inches from the closed end. If the area of the piston be 8 square inches, what force must be exerted to hold the piston at a distance of 12 inches from the closed end?

Ans. 20 lbs.

(17) An elastic hollow ball, one foot in diameter, contains air at the ordinary pressure, and at temperature 60° Fahrenheit. The temperature is increased to 100°, during which the ball increases to 13 inches in diameter. Find the change in the amount of the total internal pressure on the ball.

Ans. $\frac{1}{169}$ diminished.

(18) Two straight glass tubes of small bore, the areas of whose transverse sections are as 2 to 1, are joined in one straight line. The smaller part, which is 35 inches in length, is closed at its end, and filled with mercury. The tube is then inverted. Show that the surface of the mercury will fall 10 inches, if the barometer stands at 30 inches.

(19) A balloon filled with gas at a temperature of 65°C and barometric pressure of 30 inches, ascends to a height where the thermometer indicates 45°, and the barometer a pressure of 27 inches. Find the change in its volume, and its approximate height. Ans. ·045 nearly; 2750 ft.

(20) A thin cylindrical vessel 12 feet high floats in water with $\frac{3}{4}$ of its axis immersed. The vessel is inverted and forced down in water till it comes to a position of equilibrium, no air being allowed to escape. Find the depth of the vessel below the surface, the height of water barometer being 33 ft.

Ans. 2 ft.

(21) $p_1\ \rho_1\ t_1$, $p_2\ \rho_2\ t_2$ and $p_3\ \rho_3\ t_3$ be corresponding values of the pressure, density and temperature of the same gas in three different observations, shew that

$$t_1\left(\frac{p_2}{\rho_2}-\frac{p_3}{\rho_3}\right)+t_2\left(\frac{p_3}{\rho_3}-\frac{p_1}{\rho_1}\right)+t_3\left(\frac{p_1}{\rho_1}-\frac{p_2}{\rho_2}\right)=0.$$

(22) A barometer tube 34 inches long opens into a large basin of mercury, so that any small rise or fall of the mercury in the tube may be considered not to affect the level of the mercury in the basin. The reading of the barometer thus constructed is 31 inches, but it is found that the 19th inch is occupied by air. If this air escape to the top where there was a vacuum, shew that the reading of the barometer will fall to 28 inches.

(23) A pipe 15 feet long, closed at the upper end, is placed vertically in a tank of the same depth: the tank is then filled with water; shew that, if the height of the water barometer be 33 feet 9 inches, the water will rise 3 feet 9 inches in the pipe.

(24) A bent tube of glass, whose transverse section is one square inch, having both its branches vertical, and one much

longer than the other, is partially filled with mercury, 6 inches of the shorter arm being empty. This is now closed air-tight by a weight equal to the atmospheric pressure on a square inch : mercury is then poured into the longer branch. Find the height to which it will rise above the top of the mercury in the shorter end, the barometer standing at 30 inches. Ans. 30 in.

(25) Equal quantities of brass are employed in forming two hollow spheres ; prove that the greatest fluid pressures they will sustain are as cubes of their diameters.

(26) A heavy piston just fits into a vertical cylinder and descends by its own weight, compressing the air in the cylinder ; find the position of equilibrium ; also find how much the temperature of the enclosed air must be increased in order that the piston may be just forced out of the cylinder.

(27) The power of a hydraulic engine is conveyed by means of a metallic pipe, of which the diameter is 4 in. and thickness $\frac{1}{10}$ in. What is the greatest pressure that will not burst the pipe, the tension of the metal being 5000 lbs. on a sq. inch. Ans. 250 lbs.

CHAPTER VIII.

Determination of the Specific Gravity of Solids and Liquids.

104. It has been explained in chapter II, that the specific gravity of a substance means its relative weight, and is expressed by the ratio which the weight of a given volume of the substance bears to that of the same volume of some given substance called the *Standard substance*. For the specific gravity of all solids and liquids, distilled water at 4°C is usually taken as the standard substance. In determining therefore the specific gravity of a substance, it will be necessary (1) to determine its own weight, (2) that of an equal volume of distilled water at 4°C, a cubic inch of which weighs nearly 253 grains.

Let W be the weight of volume V of a substance whose specific gravity referred to distilled water is S ; then by Art. 27, $W = SV \times$ weight of a unit of volume of distilled water.

Similarly if W' be the weight of a volume V' of distilled water,

$W' = V' \times$ weight of a unit of volume of distilled water ;

$$\therefore \frac{W}{W'} = S.\frac{V}{V'} \quad \therefore \; S = \frac{\frac{W}{W'}}{\frac{V}{V'}}.$$

This equation suggests three different methods of determining S :

(1) By taking $V = V'$, we have $S = \frac{W}{W'}$, or the specific gravity will be expressed by the ratio of the weights. The method is carried on in practice by means of the *hydrostatic balance.*

(2) By taking $W = W'$, we shall have $S = \frac{V'}{V}$, or the specific gravity will be in the inverse ratio of the volumes. This method is pursued by means of *Nicholson's hydrometer* and the *Common hydrometer.*

(3) By taking simultaneous observations of $\frac{W}{W'}$ and $\frac{V}{V'}$. This is the principle of Sikes's hydrometer.

We will briefly describe each of these instruments, and the method of using it.

It may be remarked once for all that W and W' in the above equations should be the weights in vacuum, and not merely apparent weights in air ; but when great scientific accuracy is not required, the weight of the air displaced by the body may be neglected, and the the weight of a body in air may be taken as its absolute weight.

105. *Hydrostatic balance* :

The hydrostatic balance is an ordinary balance, having one of the scale pans provided with a hook below, from which substances immersed in a vessel of water beneath may be suspended. The hydrostatic balance is adapted to find the specific gravities of solids as well as of liquids. We will exemplify the method of observation in each case.

(1) To find the specific gravity S of a solid.

Let W = weight of the solid in vacuum. Weigh the solid in distilled water by means of the balance, and let W' be the weight. Then $W = W'$ + weight of water displaced by the solid.

Therefore W and $W-W'$ are weights of equal volumes of the solid and water;

$$\therefore\ S=\frac{W}{W-W'}.$$

If the solid be lighter than water, then a heavy body of sufficient size and weight (called *the sink*) must be attached to the solid, so that the two together may sink. The process will then be as follows:

Let x = weight of the sink in vacuum,

x' = weight of the sink in water,

w' = weight of the two together in water,

$\therefore\ W+x-w'$ = weight of the water displaced by the solid and sink together.

Also $x-x'$ = weight of water displaced by the sink alone,

$\therefore\ W-w'+x'$ = weight of water displaced by the solid,

$$\therefore\ S=\frac{W}{W-w'+x'}.$$

(2) Find the specific gravity of a liquid.

Take a solid which is specifically heavier than both the liquid and water.

Let W be the weight of the solid in vacuum, W' its weight in the liquid, and W'' its weight in water.

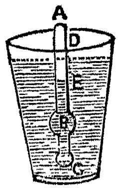
$\therefore\ W-W'$ = weight of *the liquid* displaced by the solid,

$W-W''$ = weight of *the water* displaced by the solid,

$$\therefore\ S=\frac{W-W'}{W-W''}.$$

106. *The Common Hydrometer.*

This is a glass instrument, and consists of a straight stem ending in two hollow spheres B and C. The sphere C is loaded with mercury, in order to bring the $C.\ G.$ of the instrument low enough, so that it floats with the stem vertical. The stem is graduated. When the hydrometer is immersed in a liquid, it displaces its own weight of the liquid, and by observing the volumes displaced by means of the graduation of the stem, in different liquids, their specific gravities are compared.



Thus let V be the volume of the hydrometer,

K = sectional area of the stem,

D the plane of floatation in a liquid,

and E the plane of floatation in water,

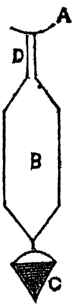
S the specific gravity of liquid.

Then $(V - K \cdot AD)S = (V - K \cdot AE) \times 1$ = weight of the hydrometer.

$$S = \frac{V - K \cdot AE}{V - K \cdot AD}.$$

This instrument is adapted for the determination of the specific gravities of liquids only.

107. *Nicholson's Hydrometer.*



This apparatus consists of a hollow metallic cylinder B, to which is fixed a loaded cone C in order to make the instrument steady.

At the top of B is a thin stem terminating in a cup. On the stem is a well defined mark D, which is called the *standard point*. The hydrometer is accompanied by a series of weights; its own weight is also carefully determined and marked. This hydrometer is adapted for determining the specific gravities of solids as well as of liquids.

(1) To find the specific gravity of a liquid.

Let W be the weight of the instrument,

W' the weight to the placed on A, in order to sink the instrument to the mark D in the liquid,

W'' the weight to be placed on A in order to sink it to D in water.

Here the volumes of the displaced liquids are the same,

$$\therefore \; W + W' = V \cdot S$$

$$\text{and } W + W'' = V \times 1$$

$$\therefore \; S = \frac{W + W'}{W + W''}.$$

(2) To determine the specific gravity of a solid.

Let W' be the weight which placed on A sinks the instrument to D in water.

Take a fragment of the solid which weighs less than W'. The weight of this solid can be determined by a scale; but is

readily determined by the hydrometer itself; for place it on A, and let W'' be placed in addition to sink the instrument to D in water. Then it is clear that

the weight of the solid fragment $= W' - W''$.

It is now necessary to determine the weight of an equal volume of water. To do this, place the solid in C, and let W''' be the weight placed on A to sink the instrument to D.

Hence $W' = W''' +$ apparent weight of the solid $= W''' + W' - W'' -$ weight of the water displaced by the solid; $\therefore$ weight of the water displaced by the solid $= W''' - W''$.

$$\therefore \text{ Specific gravity of the solid} = \frac{W' - W''}{W''' - W''}$$

Obs. 1. If the solid be lighter than water, so that it tends to float up from the pan C, it must be confind in a fine wire cage adjusted on C.

Obs. 2. When the liquid whose specific gravity is to be determined is such as corrodes metals, a glass instrument of the same construction must be used. Such an instrument is called Fahrenheit's hydrometer.

108. *Sikes's Hydrometer.*

The form of this instrument is the same as that of the common hydrometer. (*See* Fig. Art. 106). But its stem is a very thin flat bar, which makes it very sensitive. It is accompanied by a series of small weights which can be slipped over the stem above C, so as to rest on it. The stem $A\,B$ is graduated. The use of the accompanying weights is to compensate for the great sensibility of the instrument, and without them the hydrometer will be adapted only for liquids differing very little in density from water.

The volumes of the whole instrument and the accompanying weights are given.

Let the instrument float to D in a liquid with the weight W_1 at C, and to E in water with W_2 at C.

Let S be the sp. gr. of the liquid, K the sectional area of the stem, V_1 and V_2 the volumes of W_1 and W_2, W and V being the weight and the volume of the instrument.

$W + W_1 =$ weight of the liquid of volume $(V + V_1 - K.\,AD.)$ and $W + W_2 =$ weight of water of volume $(V + V_2 - K.AE)$.

$$\text{Then } S = \frac{W + W_1}{W + W_2} \times \frac{V + V_2 - KAE}{V + V_1 - K.AD}.$$

109. It is said that the hydrometer was invented by Hypatia, a lady philosopher of Alexandria in the 4th century A. D; but it was forgotten until reinvented by R. Boyle about 1675 A. D. It was primarily designed for the purpose of detecting counterfeit coins.

In the determination of densities of liquids by means of hydrometers, there is a liability to error in consequence of capillary action, or the surface tension along the line of contact of the instrument and the surface of the liquid. For this reason, neither the common hydrometer, nor Nicholson's hydrometer, gives such accurate results as the hydrostatic balance. By making the stem immersed thin enough, as in Sikes's hydrometer, this error is diminished, but cannot altogether be avoided.

Hydrometers are widely used for ascertaining the degree of consistency of various liquids in the arts and manufactures, and are known under the different names of *acidometer*, *lactometer*, *sacharometer*, *salimeter* &c. The instrument in all cases is the same, *viz* the common hydrometer, but the graduation in each is made differently to suit the purpose for which it is intended. Thus, if it is the consistency of syrup that is wanted to be ascertained, the hydrometer is successively immersed in syrups of various known consistency, *i. e.* containing known percentages of water, and the graduation is effected. For instance, Beaume's hydrometer, which is extensively used in manufactures for this purpose, is graduated thus: The instrument is immersed in pure water and the stem is marked zero at the surface. It is then immersed successively in standard solutions of pure common salt in water, containing 1, 2, 3 &c percent, by weight, of dry salt, and the stem immersed is marked 1°, 1°, 3°, &c. The graduation of Beaume's hydrometer for liquids of lesser specific gravity than water, as spirits, is thus effected: The instrument is so constructed that it floats in pure water with most of the stem above the surface. A solution containing 10 percent of pure salt in water is used to indicate the zero of the scale; and the point at which the instrument floats when immersed in distilled water at a given temperature (usually 10° R, *i. e.*, 54½° F) is marked 10°. Equal divisions are then marked off upwards along the stem up to say 50°.

Another form of hydrometer, called Twaddell's, is much used by manufacturers for liquids heavier than water. The scale is so arranged that the reading of the instrument multiplied by 5, and added to 1000, gives the sp. gr. with reference to water as 1000.

For particular information, the student is referred to special works on the subject.

110. *Specific gravity flask.*

When the substance is in a state of powder or small fragments, its specific gravity can be easily determined by means of the specific gravity flask. This is a flask with a large neck fitted with a carefully ground glass stopper. The stopper is perforated along its axis, and the bore is continued by means of a thin tube on which is a well defined mark a ; at each weighing, the flask is filled up to a with water.

Let W be the weight of the powder, and W' the weight of the flask filled with water up to a.

Pour the powder into the flask, and fill it up with water up to a, and let its weight be W''.

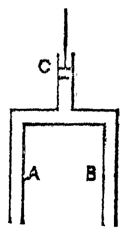
Then W'' = weight of powder + weight of flask filled with water − weight of water displaced by the powder.

$\therefore$ Weight of water displaced by the powder $= W + W' - W''$

$$\therefore \text{ Specific gravity of the powder} = \frac{W}{W + W' - W''}.$$

Specific gravities of two fluids can be easily compared by means of the specific gravity flask. For by filling the flask with each fluid up to the mark a, the weights of the same volume of the fluids may be directly ascertained, and the ratio of their weights is the ratio of their specific gravities.

111. *Hare's Hydrometer.*



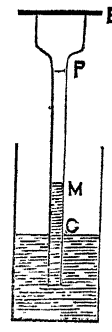
This instrument consists of two straight vertical tubes A and B both opening into a common tube C, in which a piston moves. The action of the instrument depends upon the principle of the barometer.

The legs A and B are immersed in cups containing different liquids, and a partial vacuum is produced by the motion of the piston in C. The atmospheric pressure on the surfaces of the cups forces up the liquids in A and B through heights which are inversely proportional to the specific gravities of the liquids.

This instrument though very simple in theory, does not give evry accurate results, and is seldom used.

112. *Stereometer.*

This is the name of the instrument originally constructed by Say for the purpose of determining the volumes, and therefore indirectly the specific gravities of bodies like gunpowder, which can not be immersed in liquids.



In its simplest form, it consists of a straight glass tube open at the end C, and terminating in a cup PE which can be rendred air-tight by a plate E. The substance whose volume is to be determined being placed in the cup, the tube is immersed in a vessel of mercury until the mercury reaches the well defined mark P. The plate E is then placed on the cup, and the tube raised until the surface of mercury inside it stands at M, that in the vessel being at C. MC is measured. The calculation is thus effected :

Let $MC=a$, $PM=b$, $K=$ sectional area of the tube, $h=$ height of barometer at the time of observation.

$v=$ volume of the substance,

$u=$ volume of air in the cup before the substance is placed therein.

Then by Boyle's law,

$$\frac{u-v+Kb}{u-v}=\frac{h}{h-a}.$$

From this equation v may be determined. The known weight of the substance gives its specific gravity.

The capacity of the cup may be accurately ascertained by repeating the same experiment when only air is in the cup.

113. When a mixture is formed of two or more liquids, the specific gravity of the mixture can be deduced from the known specific gravities of the components.

Let V_1, V_2, V_3 &c. be the volumes of the component liquids and S_1, S_2, S_3, &c. their specific gravities.

Let S be the specific gravity of the mixture. The volume of the mixture, unless changed by chemical action,

$$=V_1+V_2+V_3+\&c.$$

$$\therefore\ S(V_1+V_2+\ldots\ldots)=V_1S_1+V_2S_2+\ldots\ldots$$

$$\therefore\ S=\frac{V_1S_1+V_2S_2+\ldots\ldots}{V_1+V_2+\ldots\ldots\ldots}.$$

If the volume of the mixture be u, chemical action having taken place, then

$$S=\frac{V_1S_1+V_2S_2+\ldots}{u}.$$

If the weights of the component liquids be W_1, W_2, W_3 &c their volumes are $\frac{W_1}{S_1}, \frac{W_2}{S_2}$ &c., and the specific gravity of the mixture

$$S=\frac{W_1+W_2+\ldots\ldots}{\frac{W_1}{S_1}+\frac{W_2}{S_2}+\ldots\ldots}$$

A table of specific gravities of some of the common substances with reference to distilled water at $4^\circ C$, is given below.

Platinum rolled	22·069	Diamond	3·531	Mercury	13·598
Gold cast	19·258	Flint glass	3·329	Sulphuric acid	1·841
Silver ,,	10·474	Marble	2·837	Milk	1·032
Lead ,,	11·352	China porcelain	2·385	Sea water	1·026
Copper ,,	8·788	Sulphur	2·033	Olive oil	0·915
Brass ,,	8·383	Coal	1·329	Turpentine	0·870
Iron ,,	7·207	Melting ice	0·930	Alcohol	0·803
Tin ,,	7·241	Oak wood	0·845	Ether	0·723
Zinc ,,	6·861	Pine	0·657	Tar	1·015
Antimony	6·71	Cork	0·240	Vinegar	1·026

Example 1. A diamond ring weighs a grains in air and b grains in water. The specific gravity of diamond being 3·5 and that of gold 19·2, find the weight of the diamond.

Let W = weight of gold in grains,

W' = weight of diamond in grains, and u the weight of a unit of volume of water, in grains.

The weight of the water displaced by the ring $=\left(\frac{W}{19{\cdot}2u}+\frac{W'}{3{\cdot}5u}\right)u.$

$\therefore$ By the conditions of the problem

$$a=W+W'$$

$$\text{and } a-b=\frac{W}{19{\cdot}2}+\frac{W'}{3{\cdot}5}.$$

By solving the simultaneous equations, we get W'.

Example 2. King Hiero's crown consisted of an alloy of gold and silver. When immersed in a vessel just full of water, the volume of the fluid thus made to flow over was ascertained. A mass of pure gold and a mass of pure silver, each equal in weight to that of the crown, were then separately immersed in the vessel just full of water, and the volumes of the fluid which overflowed in each case were also ascertained. To determine the volume of gold and silver in the crown.

Let W = weight of the crown, and C its volume which is known,
V and V' the volumes of gold and silver in it,
S and S' the specific gravities of gold and silver respectively.
U = weight of a unit of volume of water.

$$\therefore \text{ volume of crown} = V + V' = C \ldots\ldots\ldots\ldots\text{i}$$

$$W = (VS + V'S')U \ \ldots\ldots\ldots\text{ii}$$

Also if a and β be the volumes of the gold and silver of the same weight as the crown,

$$W = aSU = \beta S'U \ldots\ldots\ldots\ldots\ldots\ldots\text{iii}$$

$$\therefore \text{ From ii and iii } \quad \frac{V}{a} + \frac{V'}{\beta} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\text{iv}$$

Hence from i and iv,

$$V = \frac{a(\beta - C)}{\beta - a}, \text{ and } V' = \frac{\beta(C - a)}{\beta - a}.$$

EXERCISES.

(1) The specific gravities of gold and silver are 19·4 and 10·5; an article composed of gold and silver weighs 979 oz. in vacuo and 890 oz. in water; compare the quantities of gold and silver in the article. Ans. 5 : 84.

(2) A solid body when immersed in distilled water weighs 9 oz.; when immersed in a particular fluid it weighs 6 oz.; and when immersed in a mixture of equal parts by weight of this fluid and distilled water, the body weighs 8 oz. Find the specific gravity of the fluid. Ans. 2.

(3) A substance which is soluble in water weighs 7 grains in air; when covered with wax, the specific gravity of which is ·96, the whole weighs 9·4 grains in air, and 3·4 in water; find the specific gravity of the substance. Ans. 2.

(4) The specific gravity of silver being 10·5 and of copper 8·9, find the relative weights of the two which must be mixed in order to form a compound which shall weigh one ninth more in air than in water. Ans. 7 : 89.

(5) A gilded iron spoon is found to have the sp. gr. 8 ; find the ratio of the volumes and weights of the metals employed, the sp. gr. of gold and iron being 19·4 and 7·8 respectively.

Ans. 1 : 57 ; 97 : 4446.

(6) The volumes of two fluids are as n : m, and their sp. gr. are S and S' ; when mixed together they lose one pth of their volume : find the sp. gr. of the mixture.

$$\text{Ans. } \frac{ns - ms'}{(p-1)(m+n)}.$$

(7) A nugget of quartz and gold weighs 12·6 oz., and its specific gravity is 7·8. The specific gravity of quartz being 2·6 and that of gold 19·4, find the weight of gold in the nugget.

Ans. 9·7 oz.

(8) A piece of cork weighs $\frac{1}{2}$ oz. in vacuo : a piece of metal weighing 6 oz. in vacuo, and $3\frac{1}{2}$ oz. in water, is attached to the cork, and the two together weigh 2 oz. in water. Find the specific gravity of the cork. Ans. ·25.

(9) A ball of India rubber, 2 inches in diameter, encloses a ball of cork 1 inch in diameter, and floats in water. The specific gravity of India rubber is ·98 and of cork is ·24 : find what proportion of the volume of the ball will float above the surface of the water. Ans. 9 : 80.

(10) In Nicholson's hydrometer, when a certain solid whose weight is 2 oz. is placed in the upper cup, a weight of $1\frac{1}{2}$ oz. must be placed in the upper cup to sink the instrument to a given depth ; when the solid is placed in the lower cup, a weight of 3 oz. must be placed in the upper cup to sink the instrument to the same depth ; find the specific gravity of the solid.

Ans. $1\frac{1}{3}$.

(11) In Nicholson's hydrometer, if the fluid be water, the substance a mixture of two metals whose specific gravities are 14 and 16, and the weights used be 16 oz., 1 oz. and 2 oz., find the quantity of each metal in the mixture.

Ans. 7 oz. and 8 oz.

(12) A common hydrometer being graduated upwards, its readings for two different fluids are x_1, x_2 ; and for mixture of equal parts of these the reading is x. Shew that the volume of a unit of length of the stem is to the volume of the whole instrument below the zero point, as $x_1 + x_2 - 2x : xx_1 + xx_2 - 2x_1x_2$.

(13) A bottle filled with water is found to weigh 5000 grams. 180 grams of powder are introduced, and the bottle is filled with water; the weight of the bottle is now found to be 575 grams. Required the sp. gr. of the powder. Ans. $1\frac{5}{7}$.

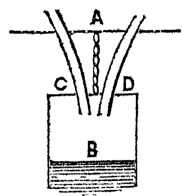
(14) A specific gravity flask is capable of holding 1000 grs. of water. 3000 grs. of a powder are introduced and the flask is filled with a liquid whose sp. gr. is ·8. The contents of the flask are found to weigh 980 grams. Required to find the sp. gr. of the powder. Ans. 2.

(15) Two liquids are mixed (1) by volume in the proportion of 1 : 4, and (2) by weight in the proportion of 4 : 1. The resulting specific gravities are 2 and 3 respectively. Find the sp. grs. of the liquids. Ans. 6 and 1; or 4 and $1\frac{1}{2}$.

CHAPTER IX.

Application of Hydrostatical Principles in the Construction of Instruments and Machines.

114. *The Diving Bell.*



A large heavy chest B open at the bottom, is suspended by means of a chain and lowered into the water. The air in the chest, being compressed, prevents the water from rising in it beyond a certain height, which depends upon the depth of the chest below the surface of the water. Seats are provided for divers in the upper part of the chest. Two pipes C and D lead to the inside, through one of which pure air is forced in from the outside, and impure air escapes through the other.

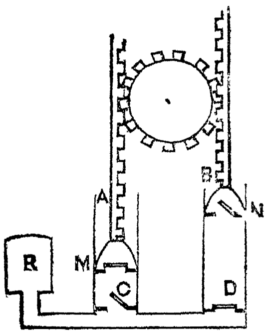
To illustrate the principle of this machine, plunge a glass tumbler into water with mouth downwards. It will be found that the surface of the water inside is much below the surface outside. This is due to the compression which the air inside undergoes.

The tension of the chain is obviously the weight of the apparatus diminished by that of the water displaced.

Diving bells were known in Europe in the sixteenth century, but were introduced into England in the reign of Charles II. The first diving bell of note was made by Dr. Hally, and was

in the form of truncated cone. Smeaton improved the apparatus. His bell was of cast iron in the form of an oblong chest, affording room for two men to work in it. It was used with success at Ramsgate. Diving bells are now being abandoned in favour of diving helmets, a more convenient and safe apparatus.

115. *Hawksbee's or the Common Air Pump.*



This consists of two cylindrical barrels A and B, into which are fitted pistons M and N, with valves opening upwards. The pistons are worked by means of a toothed wheel in such a way that when one piston ascends, the other descends. The barrels communicate by means of the valves C and D opening upwards, with a pipe leading into the receiver R, which is to be exhausted of air.

Mode of action. Suppose M to be in its lowest position, and therefore N at its highest position. Turn the wheel so that M ascends and N descends ; the valve M closes, and the valve N opens. The air in R opens the valve C, and occupies the barrel A, the valve D all the time remaining closed. The result is that at the end of the first stroke, the air in the receiver occupies the receiver and the barrel A, thereby becoming rarefied, the capacity of the thin pipe being neglected.

When the wheel is turned back, precisely the same sort of action takes place ; for M descending, the air in A is compressed, which opening the valve M, escapes out, while the valve at N remains closed during its ascent, so that the air of the receiver forces the valve D open, aud occupies the barrel B, thereby its density being further reduced.

The process is repeated till the air in the receiver is so rarefied that it can not lift the valves C and D.

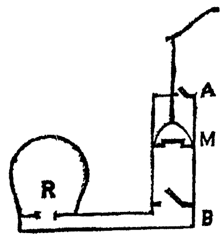
Let A be the capacity of either barrel, and R that of the receiver. Also let ρ be the density of the air in the receiver, and $\rho_1, \rho_2 \ldots\ldots \rho_n$ its densities at the end of the 1st, 2nd......nth strokes. Then by Boyle's law,

$$\frac{R}{A+R}=\frac{\rho_1}{\rho},\ \frac{R}{A+R}=\frac{\rho_2}{\rho_1}, \text{ and so on.}$$

$$\therefore \left(\frac{R}{A+R}\right)^n=\frac{\rho_n}{\rho}.$$

This gives the density of the air after n strokes of the piston. It is evident that perfect exhaustion can not be obtained by this instrument.

116. *Smeaton's Air Pump.*



This consists of a single closed barrel in which a piston M with a valve opening upwards works through an air-tight collar. At the ends of the barrel are valves A and B opening upwards. The barrel communicates with the receiver R by means of a thin pipe.

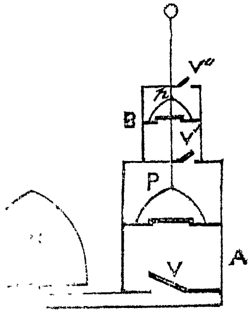
Suppose the barrel filled with atmospheric air, and the piston M at its lowest position. When the piston ascends, the pressure of the air in R opens the valve B, and it occupies the portion MB, while the valve M remains closed, and the air in MA escapes through the valve A.

When the piston descends, the valves A and C remain closed; the air in MB being compressed, opens the valve M, and passes to the upper side of the piston. When the piston ascends again, the air is expelled through A, and another portion of the air from the receiver fills the barrel. Thus the density of the air in the receiver is rapidly reduced. The advantage of this pump is, that since the valve A closes as soon as the piston begins to descend, there is no atmospheric pressure on the top of the piston, during its downward course. Hence a slight pressure of the air from beneath is capable of opening the valve M, and accordingly a greater vacuum can be obtained than in Hawksbee's pump. Another advantage is that the worker is relieved of the atmospheric pressure during a greater part of the upward stroke.

The density of the air in the receiver after n ascents of the piston can be determined precisely in the same way as in the case of Hawksbee's pump. Thus if A be the capacity of the barrel, and R that of the receiver, ρ the density of the air in the receiver at first, and ρ_n its density after n ascents up the piston, we have

$$\rho_n = \rho\left(\frac{R}{A+R}\right)^n,$$

117. *Siemen's Air Pump.*



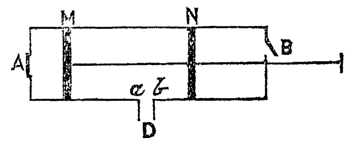
This consists of two strong cylinders A and B of equal length but different sectional areas. The smaller is attached either to the top or bottom of the larger, and they have a common axis. A is called the exhausting cylinder, and its sectional area is 3 or 4 times larger than that of B, and therefore its volume is also as many times greater. The only air passage between them is a silk valve V'. In each cylinder works a valved piston P and p, attached to the same piston-rod common to both, which passes through a stuffing box in the plate. The distance between the pistons is such that while P is in contact with the top of A, p is in contact with the top of B; and when P is in contact with the bottom of A, p is also in contact with the bottom of B. The receiver communicates with the larger cylinder A through a valve V opening inside.

Action of the pump.

As the piston P ascends from the lowest point, the elasticity of the air in R opens the valve V and fills A. The air in A is at the same time pressed through the valve V', and enters the cylinder B, in which a vacancy is simultaneously made for it by the ascent of the piston p; and in consequence of the difference of the volumes of A and B, is compressed to $\frac{1}{3}$ or $\frac{1}{4}$ of its former volume. The air in B is in the same manner pressed out through the valve V''. During the descent of the piston, the valves V, V' and V'' remain closed, and the valves in the pistons are opened by the pressure of the air in the cylinders, admitting the air in each cylinder to pass from one side to the other of the pistons as they descend, so that in the following upward stroke, the air which during the previous stroke issued from the receiver into A, is withdrawn from that into the cylinder B, while the air condensed in the latter is finally expelled.

By this arrangement, a more perfect vacuum can be obtained than by any pump previously devised. It can also be worked with greater ease, as the inequality of pressure on the two faces of the piston is not considerable; for the resisting pressure against the piston P can not rise at the termination of the stroke to more than 3 or 4 times the pressure of the air still contained in the receiver, and therefore *diminishes* as the perfection of the vacuum increases.

118. *Tait's air pump.*



This consists of a single cylinder, with two valves at the two ends, opening outwards. A spout D of breadth ab in the middle of the cylinder, communicates with the receiver to be exhausted, by means of a tube. Two solid pistons M and N are rigidly connected together by the same piston rod which acts through airtight collar. The distance between the pistons is so arranged that $MN = Aa = Bb$, so that when M is at A, N shall be at a; and when N is at B, M shall be at b. The cylinder is generally placed horizontal for convenience of working.

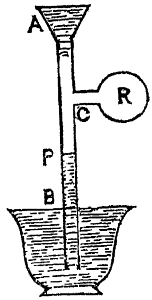
Action of the pump.

Suppose at the beginning, the pistons M and N are at A and a respectively, and the barrel filled with atmospheric air. As the rod is pulled outwards, the air in NB escapes through the valve B. The valve A remains closed, and the space AM is a vacuum, until at the end of the stroke, M arrives at b, when it is filled with air from the receiver. At the next opposite stroke, the air in AM is expelled by the piston M through the valve A, the space BN now becomes empty, until N having arrived at a, a further quantity of air from the receiver fills it; which again is expelled through B at the next stroke of the piston. In this manner a volume of air nearly equal to half that of the cylinder is driven out at each stroke of the piston.

The tension of the piston rod at any moment is the difference of the pressures on the two faces of the piston; and as this difference decreases with each stroke, the working becomes easier as the exhaustion proceeds.

119. *Sprengels' air pump.*

This instrument in its simplest form, consists of a straight glass tube open at both ends. Its legth is considerably greater than that of a barometric tube. The upper end is tightly fitted to a funnel A, and the lower end dips into a vessel of mercury B. The tube has in its side a spout C, which may be attached to a small receiver R to be exhausted of air.



Action of the pump.

Mercury is poured in A. As each drop passes the neck C, it carries down with it a quantity of air; and as no air enters the tube from either end, the vacuum caused by this means is immediately filled by air from the receiver, which expands and occupies a portion of the tube about C. In this way, the tube becomes filled with alternate small columns of air and mercury, which gradually pass out into the vessel B. The mercury which flows out may be used over again, and a continuous flow of mercury in the funnel rapidly exhausts the receiver. As the exhaustion proceeds, mercury rises in the lower part of the tube, until when the exhaustion is complete, the height BP of a continuous column of mercury in the tube is equal to that of the mercurial barometer.

By means of this simple contrivance, greater exhaustion can be attained than by any other method. The vacuum in what are called Giessler's tubes, and in globes of incandescent electric lamps, is produced in this way.

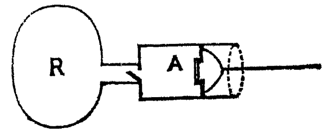
By means of an improved form of Sprengel's pump, called M. Alvergniat's *mercurial machine*, exhaustion can be effected with great rapidity.

120. *Condensing pump, or Condenser.*

A condensing pump is of the same nature as an exhausting pump, with action reversed. A strong vessel is held tightly down with screws, and by a reversed action of the valves, air is pumped *into* and not out of the vessel. (The annexed figure will clearly explain its action.)

The condensing pump is of great use in supplying fresh air to a diver or diving bell.

To find the desity of the air in the receiver after n descents of the piston :



It is evident that at each descent of the piston, atmospheric air occupying the barrel A is forced into the receiver R. Hence if ρ_1, ρ_2 ...ρ_n be the densities in the receiver after the successive descents of the piston, and ρ the density at first, we have by Boyle's law,

$$\frac{\rho_1}{\rho}=\frac{A+R}{R},\frac{\rho_2}{\rho}=\frac{2A+R}{R}\ \&c.\ \frac{\rho^n}{\rho}=\frac{nA+R}{R}.$$

121. *Manometer.*

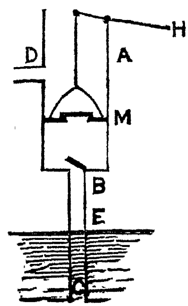
This is the general name given to any instrument for measuring the pressure of air in the receiver of an exhausting or condensing pump. There are various forms of this instrument ; the simplest and of common use are the following :

(1) *Barometer Gauge* :—This is a straight graduated tube, the upper end of which communicates with the receiver of an air-pump, the lower end being immersed in a cup of mercury. As the exhaustion proceeds, mercury rises in the tube, and the column is supported at any moment by the difference of the atmospheric pressure on the surface of the cup, and that of the air in the receiver. Thus if x be the observed height at any moment, and h the height of the barometer, the pressure of air in the receiver is equal to $g\rho(h-x)$.

(2) *Siphon Gauge* :—This is a bent tube, closed at one end and open at the other. The branches are straight and vertical. The closed branch is wholly filled with mercury which extends partly to the other. The open end being connected with the receiver of an air-pump, as the exhaustion proceeds, mercury rises in this branch and falls in the closed branch, leaving vacuum at the top. The pressure in the receiver at any moment is indicated by the difference of the heights of mercury in the two branches.

(3) *Gauge of a Condenser* :—In its simplest form, this instrument conists of a graduated glass tube, closed at one end, and the open end is connected with the receiver of the condenser. The air in the tube is separated from that of the receiver by a drop of mercury. As the condensation proceeds, the drop of mercury is pushed towards the closed end, thereby compressing the air in the tnbe. The extent of compression is indicated by the graduation in the tube.

122. *The Common or Suction pump.*



This consists of a cylinder AB communicating with a thinner cylinder or pipe BC by means of a valve opening inwards. A piston M with a valve opening upwards is worked up and down the cylinder by the handle H.

The lower part of BC rests in water which is to be pumped up ; D is a spout through which the water flows out.

Action of the pump. Suppose the piston M to be at B. At first the machine works exactly like an air-pump, the air in BC being gradually exhausted, atmospheric pressure on the surface of the reservoir forces the water to rise in BC, until at length the water rises through the valve B. At the next downward stroke of the piston, the mass of water in BM passes through the valve M on the upper side of the piston, and at the next upward stroke, it is lifted and discharged through the spout D.

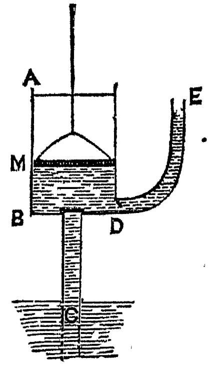
For the successful action of the pump, it is necessary that the atmospheric pressure should be capable of raising the water above the valve B; in other words, the vertical height of B above the surface of water in the reservoir must not be greater than 33 feet. In practice, it should be considerably less.

It is not theoretically necessary to make the pipe BC of less diameter than the barrel AB ; but it is practically convenient to do so.

The tension of the piston rod at any moment is the difference between the pressures on the upper and lower faces of the piston M. When the pump is in full action, the tension upwards is therefore evidently equal to $g\rho\,(EM-DM)A$, A being the area of the piston.

123. *The Forcing Pump.*

A cylinder AB communicates with the pipe BC leading into the reservoir by means of the valve B, as in the common pump. A solid piston M works up and down AB. A pipe DE communicates with AB by means of a valve at D opening upwards.



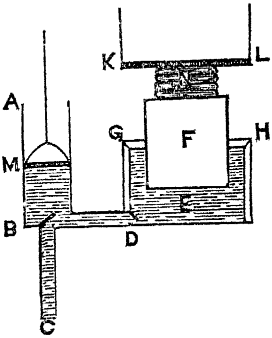
When the piston is raised from B, the valve D remains closed, the valve B opens, and the air in BC passes through it in BM. At the downward stroke, B remains closed and this air is forced out through the valve D. As the air in BC is exhausted, the water rises by the atmospheric pressure.

By continuing the process, water rises through B, and at the next downward stroke of the piston, is forced up the pipe DE. At each downward stroke, a quantity of water is thus forced up, and the surface of the water in the pipe DE rises higher. In order to avoid the intermittent action of the pump and make the stream *continuous* through the pipe at E, an *air-tight* vessel, (not shown in the figure,) is sometimes interposed between the cylinder AB and the pipe DE. The valve D opens into this vessel, and the pipe is inserted into it. As the water rises in this vessel, the air in it is compressed in its upper portion, and the reaction of this air causes a continuous flow of water through the pipe. For fuller information about these and analogous contrivances, the student is referred to special works on the subject.

124. *The Fire Engine.*

A double Forcing Pump, both pumps communicating with the same air-vessel, and worked alternately by means of a common lever, constitutes a Fire Engine. Water is forced through a pipe leading from the air-vessel, and discharged through the other end. The machine is used to extinguish fires in towns, and hence its name.

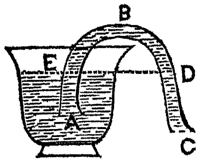
125. *Bramah's Hydraulic Press.*



$ABCD$ is a forcing pump; E is a strong chamber into which water is forced through the valve D which opens in E. F is a strong cylindrical piston working up and down in a water-tight collar through GH. N is the substance placed on F, and is pressed against a fixed platform KL, when F is raised by the forcing of the water in E. When the work is done, a plug is opened, through which the water in E flows out, and the cylinder F returns down to its former position, when it can be worked up again.

The action of the machine depends upon the principle of the equality of the transmisson of fluid pressure. For let M and F be the sectional areas of the piston M and the cylinder F. Then if a pressure P be applied to the piston M, the pressure transmitted to the cylinder F will be $F \times \frac{P}{M}$. Hence by making F many times larger than M, a small pressure P may be made to produce a considerable pressure on the cylinder F. The power of this machine is only limited by the strength of the materials of which it is formed.

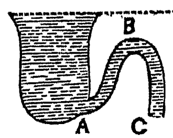
126. *The siphon.*



This is simply a bent pipe ABC, having one branch BC longer than the other AB. The pipe is filled with liquid, the ends are closed, and the branch BA is placed in a vessel containing the same liquid with which the pipe is filled. If the ends be opened at the same time, there is a continuous flow of the liquid through the end C, provided that the vertical height of B, the highest point of the siphon, above the surface E, is never greater than the column of the liquid which the atmosphere can support.

For let the horizontal surface of the liquid in the vessel meet BC in D. The pressure of the liquid at C is greater than at D, and therefore greater than the atmospheric pressure.

Hence when C is opened, the fluid flows out. This tends to cause a vacuum at B, which is however immediately filled by the liquid forced up through A by the atmospheric pressure on the surface E. So the fluid in the vessel is emptied by continuous flow through C until the surface E falls below A. The siphon is a convenient instrument in pouring the contents of one vessel into another placed on a lower level.



127. Intermittent springs can be explained on the principle of the siphon. For in the annexed figure, it is evident that there will be flow of water through C, so long as the surface of the water in the reservoir R does not fall at a lower level than B. If the surface in R rises periodically, the flow through C will be periodically intermittent.

EXERCISES.

(1) The height of the water-barometer being 34 feet, what would be the pressure of the air on a square foot inside a Diving bell at a depth of 400 feet below the surface of water?

Ans. 434000 oz.

(2) The area of the cistern of a barometer is 5 times that of the tube, and when the barometer stands at 30 inches the highest point of the tube is 2 inches above the upper surface; if a mass of air which at atmospheric density would fill one inch of the tube be admitted into the upper portion, shew that the column of mercury will be depressed 4 inches.

(3) If a diving bell be cylindrical and of length l, what is the depth of its top below the surface when half of it is occupied by water? And, if the bell remain in this position, what quantity of atmospheric air of the same temperature as the water can be forced into it, the height of water barometer being h?

Ans. $h - \frac{l}{2}$; $\frac{l(2h+l)}{2h}$.

(4) Find the greatest available length of the shorter leg of a siphon used for drawing off a fluid whose specific gravity $= 1·125$, when the mercurial barometer stands at 29 inches.

Ans. 29 ft. nearly.

(5) In Smeaton's air pump, find the position of the piston at that moment of the nth stroke when the upper valve begins to open.

Ans. Distance from the top $= a\left(\dfrac{R}{A+R}\right)^{n-1}$.

(6) Find the greatest height over which a quantity of oil, whose density is ·9, can be raised by a siphon, the height of the mercury in the barometer being 30 inches, and its density 13·6.

Ans. $37\frac{7}{9}$ ft.

(7) The legs of a siphon are at right angles to one another, and are 6 and 8 feet long. Find the greatest depth of fluid which can be drawn from a vessel by means of such an instrument. Ans. 4·8 ft.

(8) In the Common pump, the two cylinders are of equal radius and of the same height, namely $\frac{3}{8}h$, where h is the height of the water-barometer. Find the height through which the water ascends in the first stroke, the piston being supposed to ascend from the bottom of its cylinder to the top. Ans. $\frac{1}{4}h$.

(9) The area of the piston of a Fire engine is 20 times the section of the leathern pipe which conducts the water from it. The piston descends at each stroke 15 inches, and makes 4 strokes in a minute; how long will the water take in reaching a fire $\frac{1}{8}$th of a mile distant? Ans. 6·6 min.

(10) If the capacity of the cylinder in the common air pump be 10 cubic inches, and that of the receiver be 90 cubic inches, find the quantity of air at atmospheric pressures contained in the receiver after six complete strokes.

Ans. 47·7 cub. in. nearly.

(11) A free diving bell is gently forced down into still water. Show that if the weight of the bell and of the air it contains be less than the weight of the water displaced by the whole volume of the bell, it will at length arrive at a position of unstable equilibrium.

(12) If in this position of equilibrium the top of the bell were 100 feet below the surface when the water barometer stood at 30 feet, what would be the depth of the top of the bell in its position of equilibrium when the water barometer stood at 31 feet?

(13) If the weight of a diving bell be 16 maunds and the depth of the surface of the water within the bell below the outer surface be equal to the height of the water-barometer, show that the tension of the chain is 7 maunds, if the specific gravity of the material of the bell be 8.

(14) The section of each cylinder of a Hawksbee's air-pump is 1 square inch in area, and each piston works up and down through a range of 4 inches from the bottom. The receiver contains 200 cubic inches of air, barometer standing at 32 inches. At what height will the barometer stand after 10 strokes of the piston? Ans. 26·3 in. nearly.

(15) A diving bell is made of a substance whose specific gravity is 4, and its interior will contain a quantity of water, whose weight is twice that of the bell. If the bell be lowered in water till the tension of the rope is half the weight of the bell, prove that the density of the air inside will be 8 times that of the atmosphere.

(16) A cylindrical diving bell is immersed in water. If a be the height of the bell, x that of the portion unoccupied by air, h the height of the water barometer, and the temperature increased from τ to $\tau + t$, prove that the surface of the water will be depressed in the bell through a space

$$\frac{hax}{ha+x^2} \cdot \frac{at}{1+a\tau} \text{ nearly.}$$

(17) If the area of the section of a forcing pump be 20 square inches, and the range of the piston be 2 feet, and if the surface of the water in the reservoir into which it is forced be uniformly 50 feet above the piston at its highest point, find the force required to hold the piston (i) at its highest point; and (ii) at its lowest point.

(18) If the volume of the receiver of a condenser be m times that of the barrel, show that during the $(n+1)$th downward stroke, the valve of the receiver will open when the piston has descended $\frac{n}{m+n}$th the length of the barrel.

(19) In Hawksbee's air pump, the volume of the receiver is n times that of the barrel. It is found that after n strokes a body which in air weighed n lbs, weighs $(n+1)$ lbs in the receiver. Determine the weight of the body in vacuum.

Ans. $\frac{(n+1)^{n+1} - n^{n+1}}{(n+1)^n - n^n}$ lbs.

(20) A bladder $\frac{1}{8}$th filled with atmospheric air, is placed under the receiver of an air pump, the capacity of the receiver being twice that of the bladder. Shew that it will be fully distended before the completion of the 6th stroke.

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UNIVERSITY PAPERS.

PROBLEMS SET IN THE CALCUTTA UNIVERSITY EXAMINATIONS FROM 1860—1884.

SPECIFIC GRAVITY.

N. B.—It is assumed that a cub. ft. of distilled water weighs 1000 oz. and the acceleration of gavity is 32 ft. in a second.

1. The weights of 2 balls are as 11 : 3 and their sp. grs. as 3 : 2 ; compare their diameters. Ans. $\sqrt[3]{66} : 3$.

2. If one second be the unit of time, and the unit of volume of the standard substance weigh 16 lbs, determine the unit of length in order that the formula $W=g\rho V$ may give the weight in pounds. Ans. 2 ft nearly.

3. If one second and one yard be the units of time and length, compare the density of the standard substance in the equation $W=g\rho V$, with that of water, the unit of weight being 1000 oz. Ans. 32 : 81.

4. If one foot be the unit of length, what must be the unit of time in order that the unit of weight in $W=g\rho V$ may be the 20 oz., water being the standard substance? Ans. 1·25 secds.

5. If the units of weight, length and time be 1 lb. 1 ft. and 1 scd. respectively, compare the densities of the standard substances in the equations $W=gV$ and $W=g\rho V$. Ans. 1 : 32.

6. The unit of length being 2 ft. find the units of time so that the units of weight in the formula $W=SV$ and $W=g\rho V$ may be the same. Ans. ·25. seds.

7. If 5 cub. ft. of a substance weigh 250 lbs, what is its sp. gravity? Ans. ·8.

8. If m cub. inches of one substance weigh as much as n cub. inches of another substance, compare their sp. grs. Ans. $n. m.$

9. Hiero's crown consisted of gold and silver, and was found to lose $\frac{1}{14}$ of its weight in water ; equal weights of gold and silver were weighed in water and found to lose $\frac{4}{77}$ and $\frac{2}{21}$

of their weights respectively. Find the proportion of gold and silver in the crown. Ans. 11 : 9.

10. A body weighs 4 oz. in vacuum, another body which weighs 3 oz. in water is attached to it, and the united mass weighs $2\frac{1}{2}$ oz. in water. Find the sp. gr. of the former body. Ans. ·84.

11. If W_1 W_2 W_3 be the apparent weights of a substance in three fluids whose sp. grs. are ρ_1 ρ_2 and ρ_3, shew that

$$W_1(\rho_2-\rho_3)+W_2(\rho_3-\rho_1)+W_3(\rho_1-\rho_2)=0.$$

12. Three fluids whose sp. grs. are 4, 5 and 7 are mixed together in the proportion of 5, 7, and 9 volumes respectively. Find the sp. gr. of the mixture. Ans. 5·6.

13. A cone floats with its axis vertical and vertex downwards in 2 fluids ρ and ρ'; $\frac{2}{3}$rds of its axis are immersed in the heavier fluid ρ'. Find the sp. gr. of the cone.

Ans. $\frac{1}{27}(8\rho'+19\rho)$.

14. The sp. gr. of sea water is 1·027; what proportion of fresh water must be added to a quantity of sea water that the sp. gr. of the mixture may be 1·009? Ans. 2 : 1.

15. The sp. gr. of a gilded iron spoon is 8; compare the volumes and weights of the iron and gold employed, their sp. grs. being 7·8 and 19·4 respectively.

Ans. 57 : 1 and 2223 : 97.

16. A substance weighs 20 lbs. in water and 160 lbs. out of water. Find its sp. gr. Ans. $1\frac{1}{7}$.

17. Each division of the stem of a common hydrometer contains $\frac{1}{m}$th part of the volume of the hydrometer. The hydrometer floats in two fluids with x and y divisions of the stem out of them respectively. Compare the sp. grs. of the fluids. Ans. $m-y : m-x$.

18. A body weighs 10 oz. in water; another body weighs 15 oz. in air (sp. gr. = ·0013). The two connected together weigh 11 oz. in water; find the sp. gr. of the latter body.

Ans. 1·0713.

19. When equal volumes of two substances are mixed, the compound has the sp. gr. = 9; when they are mixed in equal weights, the sp. gr. of the compound = $8\frac{8}{9}$. Find the sp. grs. of the substances. Ans. 10 and 8.

PRESSURE.

20. A thin circular tube is placed in a vertical plane. Equal volumes of 2 fluids each occupying an arc subtending angle α at the center are poured into the tube. Find the position of equilibrium. Ans. If θ be inclination of the raidius

$$\text{at the common surface, } \tan\theta = \frac{\rho - \rho'}{\rho + \rho'} \tan\frac{\alpha}{2}.$$

In the last question, if the volumes of the fluids occupy arcs subtending angles α and β at the center, find the position of equilibrum.

$$\text{Ans. } \tan\theta = 2\cdot\frac{\rho\sin^2\frac{\alpha}{2} - \rho'\sin^2\frac{\beta}{2}}{\rho\sin\alpha + \rho'\sin\beta}.$$

21. In two homogeneous liquids the pressures at the depths of 3 and 4 inches respectively are the same. Compare the pressures at the depths of 12 and and 16 inches respectively.

Ans. equal.

22. The pressure of a fluid, not exposed to atmospheric pressure, is 15 lbs. on a square inch at a depth of 20 ft.; what is the pressure at a depth of 12 feet? Ans. 9 lbs.

23. A trough contains mercury to the depth of 3 inches. Find the pressure on the horizontal base, the area of which is 72 sq. inches, the density of mercury being 13·6. Ans. 1700 oz.

WHOLE PRESSURE.

24. An isosceles triangle is vertically immersed in a fluid, (1) with its base in the surface, (2) with its vertex in the surface; compare the pressures on the area. Ans. 2 : 1.

25. A square is just immersed vertically in a fluid, (1) with a side vertical, (2) with a diagonal vertical; compare the pressures on the area. Ans. 1 : $\sqrt{2}$.

26. Two spheres whose radii are 3 and 5 respectively are just immersed in a fluid; compare the pressures on their surfaces.

Ans. 27 : 125.

27. A cubical box is filled with fluid and held with a diagonal vertical; compare the pressures on its upper and lower sides. Ans. 1 : 2.

28. An isosceles triangle is just immersed in a fluid with its vertex in the surface. Divide it by a line parallel to the base, so that the pressures on the upper and lower parts may be as 1 : 7.

Ans. The line bisects the sides.

29. A rectangle is immersed in a fluid with a side in the surface. Divide it into 12 horizontal strips on which the pressures may be equal.

30. A cylinder is immersed in a fluid with its axis vertical. Divide it by horizontal planes into n parts such that the pressures on the annuli may be equal.

Ans. The depths of the dividing planes below the surface are $\sqrt{\frac{1}{n}}$, $\sqrt{\frac{2}{n}}$, $\sqrt{\frac{3}{n}}$ &c. parts of the axis.

31. A right cone, its axis being vertical, is immersed (1) with its base, (2) with vertex, in the surface; compare the pressures on the curved surface. Ans. 1 : 2.

32. A right conical vessel, vertical angle being 60°, filled with fluid, rests with its base horizontal and vertex upwards. Compare the pressures on the concave surface and the base. Show that the ratio of these pressures cannot for any cone be less than 2 : 3. Ans. 4 : 3.

Floating Bodies.

33. An iron cylinder $8\frac{1}{2}$ inches long (sp. gr. $=7\cdot2$), floats vertically in mercury whose sp. gr. $=13\cdot56$. Find the portion immersed. Ans. $4\frac{1}{2}$ in. nearly.

34. A hollow sphere, whose internal diameter is 2 ft., just floats in water; find its thickness, the sp. gr. of copper $=8\cdot79$.
Ans. $\cdot49$ inch.

35. A spherical shell of iron $\frac{1}{10}$ inch thick is filled with alcohol (sp. gr. $=\cdot8$). Find its internal radius that it may just float in water. Sp. gr. of iron $=7\cdot2$. Ans. $9\cdot4$ ins.

36. A cylinder when floated with axis vertical in 2 fluids (ρ, ρ'), is found to sink $\frac{1}{m}$th and $\frac{1}{n}$th part of its axis respectively. Find what part of its axis will sink in a mixture composed of V parts of the first and V' parts of the second.

Ans. $\dfrac{V+V'}{mV+nV'}$.

37. A square lamina (sp. gr. ρ) floats with one angle immersed. Show that if ρ lie between $\frac{8}{32}$ and $\frac{9}{32}$, there will be 12 different positions of equilibrium.

38. A cube floats in water with a face horizontal; a body weighing 3000 oz. placed upon it makes it sink through an inch. Find the size of the cube. Ans. Edge $=6$ ft.

39. A solid sphere floats in a fluid with $\frac{3}{4}$ of its volume above the surface; when another sphere half as large again is attached to it by a string, the two together float completely immersed. Compare the sp. grs. of the spheres.

Ans. 6 : 1.

40. An iceberg floats in sea-water; find the ratio of the part out of water to the part immersed, the sp. grs. of ice and sea water being ·925 and 1·025 raspectively.

Ans. 4 : 37.

41. Two small pieces of cork (sp. gr. = ·25), the volume of one being 3 times that of the other, are connected by a string 3 ft. long passing over a small fixed pully at the bottom of a vessel of water 2 ft. deep. Find the position of equilibrium.

Ans. The larger piece is half immersed.

Air and Gas.

42. A straight tube 20 feet long is $\frac{1}{2}$ filled with water, and then inverted, the open end being placed on the surface of a vessel of water; no air is allowed to escape. Find how high will the water stand in the tube, the height of water barometer being 33 ft. Ans. 7·2 ft. nearly.

43. A cylindrical tube 37 inches long and closed at one end is filled with $21\frac{2}{3}$ inches of mercury, and 14 inches of water, the rest being occupied by air. Find the depth to which the mercury would subside after immersing the open end of the tube in a basin of mercury. Ans. 19 inches nearly.

44. A vertical cylinder 2 ft. long, and of diameter 3 inches, contains atmospheric air; a tight fitting piston weighing 5 lbs. is placed on the open end. Find the position where the piston will rest. Ans. 1·1 inch from top.

45. A perfectly elastic bag has forced into it air sufficient to fill 25 bags of the same size. To what depth must it be sunk in water that it may return to its natural size, the height of water barometer being 34 feet. Ans. 136 fathoms.

46. A barometer 33 inches long contains some air at the top. Its reading is 29 when a true barometer indicates 30; what is its reading when the true barometer reads 25?

Ans. $24\frac{1}{2}$ nearly.

47. An imperfect barometer, *i. e.* containing air at its upper portion, of length l, stands at a when the true barometer stands at b. At what height does the latter stand when the former stands at c?

Ans. $c + \frac{l-a}{l-c}(b-a)$.

48. Two imperfect barometers of the same length l, at a given moment, reads h h' respectively, and at another moment, k and k' respectively. Determine the quantity of air at atmospheric pressure left in each, the temperature during the interval remaining constant.

Ans. The portions of the tubes occupied are respectively

$$\frac{\frac{h'}{h}(l+h-h')-\frac{k'}{k}(l+k-k')}{\frac{h'(l-h')}{h(l-h)}-\frac{k'(l-k')}{k(l-k)}} \text{ and } \frac{\frac{h}{h'}(l+h'-a)-\frac{k}{k'}(l+k'-k)}{\frac{h(l-h)}{h'(l-h')}-\frac{k(l-k)}{k'(l-k')}}.$$

49. A Fahrenheit thermometer indicates a temperature of 68; what are the corresponding readings in the centigrade and Reaumer's scales?

Ans. 16, 20.

50. The tube of a thermometer is $\frac{1}{10}$ inch in diameter, and the distance of the boiling and freezing points is 7 inches. Find the volume of the bulb and the stem below the freezing point, assuming the expansion of mercury to be ·0001 for 1° F.

Ans. 3·054 cb. in.

51. The simultaneous readings of two mercurial thermometers which are differently graduated, are at one moment τ_1 and τ_2, and at another moment t_1 and t_2 respectively. What will be the reading of the latter when the former indicates $n°$?

Ans. $\frac{\tau_1 \sim \tau_2}{t_1 \sim t_2} \cdot n$.

Machines.

52. In Smeaton's air pump, the density of air in the receiver is reduced to one fourth its original value at the end of the 3rd stroke; compare the capacities of the receiver and the barrel.

Ans. 25 : 11 nearly.

53. A body when placed under the receiver of an air pump weighs w_1, and after n strokes of the piston weighs w_2; find the weight of the body in vacuum, the ratio of the capacities of the barrel and receiver being k.

Ans. $\frac{w_2(1+k)^n - w_1}{(1+k)^n - 1}$.

54. The range of the piston in an air-pump is a; find the depth at which the piston valve will open on the rth descent of the piston; A and B being the capacities of the receiver and barrel.

Ans. $a\left\{1-\left(\frac{A}{A+B}\right)\right\}$.

55. A forcing pump is provided with a cylindrical air-vessel of height c, and sectional area A, through which rises a pipe of section k. The section of the piston is also A and its range is l. If when the pump commences working, the water be just below the valve in the air-vessel, after how many strokes of the piston will the water rise to a height h of the water-barometer above the bottom of the air-vessel.

$$\text{Ans. } \frac{2ah + (A-a)\{c + 2h - \sqrt{c^2 + 4h^2}\}}{2Ac}.$$

56. A condenser communicates with a Smeaton's air pump through its upper valve; the capacity of the cylinder is half that of either receiver; compare the pressures of air in the receivers after 2 ascents and descents of the piston.

Ans. 33 : 8.

MISCELLANEOUS PROBLEMS SELECTED FROM THE DEGREE AND B. C. E. EXAMINATIONS OF THE BOMBAY AND MADRAS UNIVERSITIES, from 1879—1885.

1. Three fluids of densities P, 2P, 3P respectively fill a semicircular tube whose bounding diameter is horizontal; prove that the depth of one of the common surfaces is double that of the other.

2. A cube floats in water with one vertex below and three in the surface of the water. Determine the sp. gr. of the cube.

Ans. $\frac{2}{3}$.

3. A rectangular lamina floats vertically in water with one diagonal in the surface and a weight attached to the angular point below the surface; find the sp. gr. of the lamina.

Ans. $\frac{1}{3}$.

4. A solid right cone whose angle is 60° is immersed in a liquid with its vertex in the surface and axis vertical; prove that the whole pressure on the curved surface and base $= 7 \times$ resultant pressure.

5. The capacity of the receiver of a *condenser* being 30 times that of the barrel, and the length of the guage 20 inches, determine the position of the globule of mercury after 12 strokes.

Ans. $14\frac{2}{7}$.

6. Two vessels contain air at the same pressure, but at different temperaturs t and t', where t' is greater than t; the

the temperature of each being raised by the same amount, find which has its pressure most increased.

Ans. The smaller vessel.

7. If the units of weight, length and time be 1 lb. 1 yd. and $\frac{1}{2}$ scd., compare the densities of the standards in the equations $W=SV$ and $W=g\rho V$. Ans. 3 : 8.

8. A certain substance floats in water with $\frac{3}{4}$ths of its volume immersed, and in oil with $\frac{4}{5}$ths immersed; what is the sp. gr. of oil? Ans. $\frac{15}{16}$.

9. A bubble ascends through water, and at the depth of 231 ft. its diameter is $\frac{1}{2}$ inch; find its diameter when it reaches the free surface, the bubble being always spherical and the water-barometer registering 33 ft. Ans. one inch.

10. A solid cylinder and a solid hemisphere which have their bases equal are united base to base, and the solid thus formed floats in a fluid with its spherical surface partly immersed; find the ratio of the height of the cylinder to the radius of the hemisphere in order that the equilibrum may be neutral.

Ans. $1 : \sqrt{2}$.

11. Find the weight of a Nicholson's Hydrometer when it sinks alone as deep in rectified spirit of sp. gr. ·8 as in water when loaded with 100 grains. Ans 400 grs.

12. Equal weights of two fluids of which the densities are ρ and $\frac{1}{2}\rho$ are mixed together, and one-sixth of the whole volume is lost; find the density of the resulting fluid.

Ans. $\frac{9}{10}\rho$.

13. A vertical cylinder contains equal quantities of 2 liquids; compare their densities when the whole pressures of the two liquids on the curved surface of the cylinder are in the ratio of 1 : 3.

Ans. Equal.

14. What must be the thickness of a right-angled cone of copper, the inner diameter of which is 20 inches, so that it may just float with its upper edge level with the surface of the fluid; the sp. grs. of the copper and fluid being as 9 : 1, and the interior and exterior surfaces having a common base.

Ans. ·4 inches nearly.

15. What weight of water must be added to a pound of fluid whose sp. gr. is $\frac{1}{2}$, in order that the sp. gr. of the mixture may be $\frac{3}{4}$?

Ans. 2 lbs.

16. A barometer is suspended in water by a string attached to its upper end so that a portion of the string is immersed; find the height of the mercury and the tension of the string.

17. If the reservoir of the water-works of a town be 100 ft. above the level of a street, find the force necessary to restrain the plug in a pipe of which the section is a square inch at a height of 82 feet above the street.

18. A rhombus is just immersed in liquid with a diagonal vertical, and from the highest point lines are drawn dividing the figure into 3 equal parts ; shew that the pressure on the middle part is to that on either of the side parts as 11 : 8.

19. A triangle ABC floats in water with its plane vertical, the angle B being in the surface, and the angle A not immersed ; find the density of the triangle in terms of its sides.

Ans. $\dfrac{a^2+b^2-c^2}{2b^2}$.

20. What change of atmospheric pressure on the sq. in. is indicated by a fall of 1·1 in. in a barometer, the density of mercury being 13 ? Ans. 8·27 oz. nearly.

21. If the capacity of the receiver of a *condenser* be 10 times that of the barrel, after how many descents of the piston will the force of the compressed air be doubled ?

Ans. 10.

22. If 27 oz. be the unit of weight, 1 yard the unit of length, and 10 seconds the unit of time, compare the density of the standard substance with that of water in $W=g\rho V$.

Ans. 16 : 15.

23. A rhombus is just immersed in liquid with a diagonal horizontal ; compare the pressures upon the parts into which this diagonal divides it. Ans. 1 : 2.

24. The weight of the unimmersed portion of a body floating in water being given, find the sp. gr. of the body in order that its volume may be the least possible. Ans. $\frac{1}{2}$.

25. The density of mercury being 13 568, what must be the length of a water barometer inclined to horizon at 30°, when the mercurial barometer registers 30·5 inches ?

Ans. 837·6 in. nearly.

26. A bladder filled to $\frac{1}{8}$th of its extent with atmospheric air, is placed under the receiver of an air pump, the capacity of the receiver being twice that of the barrel ; after how many strokes will it be fully distended ? Ans. 6 nearly.

27. A piece of wood weighs 12 lbs, and when attached to 22 lbs of lead and immersed in water, the whole weighs 8 lbs ; the sp. gr. of lead being 11, determine the sp. gr. of the wood.

Ans. ·5.

28. The sp. gr. of a fluid is 1·56, find the depth of a point at which the pressure is 12090 oz, a cubic foot of water weighing 1000 oz.

Ans. 7·75 ft.

29. A pyramid with a square base, and the sides of which are equilateral triangles, is placed on a horizontal plane and filled with fluid through an orifice in the vertex. Find the pressure on one of the sides in terms of the weight of the fluid.

Ans. $\frac{\sqrt{3}}{2}$.

If the pyramid have no base, show that its least weight consistent with its not being raised from the plane is twice the weight of the fluid.

30. A solid cone and a solid hemisphere with equal bases are united base to base, and the compound solid floats in water with the spherical surface partly immersed in a position of neutral equilibrium; find the height of the cone.

Ans. $\sqrt{3}.r$.

31. A bubble of air $\frac{1}{2}$ inch long is confined at the bottom of a narrow bent tube of uniform bore by two columns of mercury each 1 inch in height, and an additional inch of mercury is poured into one tube; find the difference in level of the two columns.

Ans. $\frac{1}{2}$ in.

32. A substance weighs 12·5 grammes in water, and 17·3 grammes in oil; find the sp. gr. of the oil.

Ans. ·854 nearly.

33. Find the volume of the balloon, if the weight of the collapsed envolope, car and aeronaut be 500 lbs, (occupying 10 cub. ft.), the weight of 1 cub. ft. of air being ·081 lb, and that of 1 cub. ft. of the gas with which the balloon is filled, ·0054 lb.

Ans. 6603 cub. ft. nearly.

34. A uniform tube is bent into the form of a parabola, and placed with its axis vertical and vertex downwards; two fluids of densities ρ and ρ' are poured into it so that r and r' are the distances of the two free surfaces from the focus; shew that the distance of the common surface from the focus $= \dfrac{r\rho - r'\rho'}{\rho - \rho'}$.

35. Find the unit of time, when 2 ft is the unit of length, in order that the units of weight in $W = g\rho V$ and $W = SV$ may be equal, the standard substance being the same in both.

Ans. $\frac{1}{4}$ Scd.

36. Equal volumes of oil and alcohol (sp. gr ·915 and ·795 respectively) are poured into a circular tube so as to fill half a circle; shew that the common surface rests at a point whose distance from the lowest point is $\tan^{-1}\frac{4}{57}$.

37. A cylindrical vessel of height h, closed at the base, is formed of stones held together by two strings, which serve as hoops, and is filled with fluid; the strings being at equal distances a from the top and the bottom of the vessel, compare their tensions. Ans. $1^2 : 2^2\ 3^2$ &c.

Find how much of the fluid must be withdrawn from the cylinder in order that the tension of the upper string may vanish.

38. A body immersed in a fluid is balanced by a weight p, to which it is attached by a string passing over a fixed pully; and when half immersed, is balanced in the same manner by a weight $2p$. Compare the densities of the body and fluid, and determine the nature of equilibrium in each case. Ans. 3 : 2.

39. What is the limit to the density of the air in the receiver of a *condenser*, if the distances of the highest and lowest range of the piston from the bottom of the cylinder be a and a'?

Ans. $\dfrac{a-a'}{a'}\rho$.

40. A closed air tight cylinder of height $2a$ is half full of water, and half full of air at the atmospheric pressure, which is equal to that of a column h of water, without letting any air escape; water is introduced so as to fill an additional height k of the cylinder, and the pressure on the base is thereby doubled; prove that $k = a + h - \sqrt{h(a+h)}$.

41. A piece of wood weighing an ounce in air has attached to it a peice of metal weighing 4 oz. in air and 2 oz. in water, and the two together weigh an ounce in water; what is the sp. gr. of the wood? Ans. ·5

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1. A cylindrical vessel is filled with two fluids of equal volumes. The pressure on the lower half of the surface is 5 times that on the upper half. Compare the densities of the fluids.

Ans. 1 : 3.

2. A triangular lamina ABC, right angled at C, floats in a fluid of $\frac{5}{4}$ths of its sp. gr. with C just on the surface of the fluid. Shew that one of the sides containing the right angle is double the other.

3. A piston without weight fits into a vertical cylinder closed at its base and filled with air, and is initially at the top of the cylinder. If water be gently poured on the top of the piston, prove that the upper surface of water will be lowest when the depth of the water is $2h$, where h is the height of the water-barometer, and $9h$ the length of the cylinder.

4. A cubical vessel whose sides are vertical, is filled with two liquids that do not mix, the density of the one being double that of the other. If the pressure on the vertical side be to the pressure on the base as $\sqrt{2}-1 : 1$, find the distance of the common surface from the top of the vessel.

Ans. $2-\sqrt{2}$ of an edge.

5. A heavy body floats between 3 liquids, V_1, V_2, V_3 being the volumes of the portions of the body in the 3 liquids respectively. If P_1, P_2, P_3 be the forces necessary to keep the body at rest when *entirely* immersed in the 3 liquids respectively, prove that $P_1V_1+P_2V_2=P_3V_3$.

6. If 26 oz. of copper lose 3 oz. in water and 7 oz. of zinc lose 1 oz., and if an alloy of copper and zinc weighing 80 oz. lose 10 oz, find the ratio of the weights of copper and zinc in the alloy.

Ans. 7 : 13.

7. A tube of small bore in the form of an ellipse, whose major axis is double the minor axis is filled with equal volumes of two liquids that do not mix, their densities being as 5 : 1. At what angle is the minor axis inclined to the vertical, when the free surfaces of the fluids are the extremities of the major axis?

Ans. $\tan^{-1}\frac{1}{3}$.

8. A solid hemisphere of radius a, completely immersed in a fluid, is held with its base upwards so that the center of its base is at a depth $2a$ below the surface of the fluid. If the resultant horizontal and vertical pressures on the curved surface are equal, the base must be inclined at $\frac{1}{2}\sin^{-1}\frac{8}{9}$ to the horizon.

9. A solid cone whose vertical angle is $\cos^{-1}\frac{1}{4}$ floats in a liquid with its vertex above the surface, and its base just touching the surface. Prove that

Density of cone : density of liquid :: 7 : 8.

10. A conical vessel of glass is inverted and plunged into water. By inclining the vessel, a quantity of air is allowed to escape out of it. The cone is then held vertically with open end immersed, and raised until one-fourth only of its axis is below the surface of the water. If the surface of water within the cone bisects the axis, shew that $\frac{15}{16}$th of the air originally in the glass must have escaped, $2h$ being the length of axis, and h the height of the water barometer.

11. The length of the tube of a barometer is 33 inches. Some air has got into the space above mercury, and in consequence the mercury stands at 29 inches when it would stand at 31 inches if the instrument was good. What is the true reading, when the false reading is 30 inches ? Ans. $32\frac{2}{3}$ in.

12. In the suction pump, the range of the piston is a, and the vertical distance between the highest level of the piston and the surface of the water is b. If c be the height of the water barometer, prove that $4ac$ must be greater than b^2.

13. The receiver of a Smeaton's air-pump is twice as large as the cylinder, whose transverse section is 45·4 sq. inches. Shew that the tension of the piston rod, when the highest valve just begins to open in the fifth ascent of the piston, is 585lbs, taking the pressure of the atmosphere on a square inch to be 15lbs, and the valve to be without weight.

14. A triangle is immersed vertically in a liquid with its vertex downwards and base parallel to the surface of the liquid, and the triangle is bisected by a line parallel to the base. If the whole pressure on the lower half of the triangle be double that on the upper half, find the depth of the base of the triangle below the surface of the liquid. Ans. $\frac{1}{3}(5-3\sqrt{2})h$.

15. Three liquids that do not mix fill a semicircular tube whose bounding diameter is horizontal. If the density of the heaviest be equal to the sum of those of the other two, the length of the heaviest subtend an angle of 90°, and the lengths of the other two subtend 30° and 60°, at the center of the circle, find the ratio of the densities. Ans. $1 : \sqrt{3} : 1+\sqrt{3}$.

16. A heavy triangle ABC whose sides AB and BC are equal, is suspended with its plane vertical by 2 vertical strings from the points A and B, and rests with its vertex A in the surface of a liquid with half its area immersed. Compare the tensions of the two strings. Ans. 1 : 2.

17. A piece of metal weighs in vacuum 200 grains more than in water, and 160 grs. more than in spirit. Find the sp. gr. of the spirit. Ans. 1·25.

18. Two thermometers are differently graduated. One of them denotes 2 particular temperatures by $a°$ and $b°$, and the other by $c°$ and $d°$. What will the latter indicate when the former indicates $n°$?

19. A cylinder full of common air is closed at the top and bottom by 2 air tight pistons, moving freely in it. The cylinder is immersed vertically in water till its top is just on a level with the surface of the water. If the weight of the upper piston = weight of a cylinder of water of base equal to the base of the

given cylinder. and altitude equal to the height of the water barometer, and if the sum of the weights of the pistons $= \frac{1}{2}$ weight of water which the given cylinder could hold; find the distances of the 2 pistons from the upper end of the cylinder.

Ans. 0 and $\frac{1}{2}l$, respectively.

20. In a curved tube of uniform bore placed in a vertical plane with its extremities upwards, water is poured until the depth is equal to the diameter of the tube. If now a quantity of oil be poured in at either end, how will the two liquids arrange themselves?

21. A solid hemisphere is immersed so that the surface of the liquid is a tangent plane to the curved surface, and the pressure on the curved surface is to that on the circular base as 3 : 2. Find the inclination of the axis of the hemisphere to the vertical. Given that the area of the curved surface is twice that of the circular base, and that its centre of gravity bisects the axis. Ans. 60°.

22. A plane figure composed of an isosceles triangle (whose height h is equal to its base) and a square described on the base, is immersed in water with its plane vertical, the vertex of the triangle on the surface, and its base horizontal. Find the center of pressure. Ans. $\frac{31}{22}h$.

23. An iron sphere, radius $\frac{1}{4}h$, fits in a smooth horizontal cylinder with its center at a distance $4h$ from one end of the cylinder. This end is then closed, the other remaining open. Assuming the sp. gr. of iron to be to that of mercury as 4 : 7, compare the angles through which the cylinder must be tilted about its extremities, in order that the center of the sphere may move through a distance equal to its own diameter on either side of its horizonal position; and shew that the sines of these angles are as 10 : 13, h being the height of mercurial barometer.

24. In a mixture of 2 liquids whose sp. grs. are σ, σ' a body whose sp, gr is ρ loses $\frac{1}{n}$th of its weight. In what volumetric proportions are the liquids mixed? Ans. $\rho - n\sigma' : n\sigma - \rho$.

25. Equal volumes of two substances A and B being suspended from the extremities of a weightless rod 18 inches long, balance about a certain point in it. If they are both immersed in water, the fulcrum must be shifted one inch towards B to maintain equilibrium. If the sp. gr. of A be $2\frac{7}{9}$, find that of B.

Ans. $7\frac{2}{9}$.

26. A cubical vessel with its base horizontal and sides vertical is filled with equal volumes of n fluids, which do not mix, of densities ρ, 2ρ, 3ρ, &c. Compare the whole pressure on the base with that against one of the vertical sides.

Ans. $6n : 2n+1$

27. A piece of wood which weighs 57 lbs. in vacuó, is attached to a bar of silver weighing 42 lbs, and the two together weigh 38 lbs. in water. If the sp. gr. of silver be 10 5, find the sp. gr. of the wood. Ans. 1.

28. A hollow cylinder of length $\frac{8}{7}h$, open at the top and closed at the bottom, is immersed mouth downwards in water. By inclining the vessel a certain quantity of air is allowed to escape. The cylinder is then held vertically with the open end immersed until $\frac{1}{4}$th only of its length is below the surface, when the water stands within the cylinder at an altitude $\frac{1}{8}h$, h being the height of the water barometer at the time. Find how much common air was allowed to escape.

Ans. $\frac{11}{14}h$.

29. A hollow cone, height h and radius r, whose axis is vertical, is filled with fluid. Find the ratio of the whole pressure to the resultant pressure on the surface of the cone.

Ans. $\sqrt{h^2+r^2} : r$.

30. A man, whose volume is 2 cub. ft. 432 cub. inches, and whose sp. gr. = 1 1, is $\frac{4}{5}$ths immersed in water. With one hand he holds on to a vertical rope supported above, and so situated that the perpendicular let fall from the man's center of gravity on the direction of its action is 8 inches. Find the tension of the rope, and the condition of equilibrium. Ans. 675 oz.

31. At temperature $0°F$, a certain thermometer is marked 123°, and at freezing point, it is marked 99°. Assuming that the same system of graduation is maintained throughout, how would it be marked at boiling point? Ans. -36.

32. If the radii of the two cylinders in Bramah's press be as 4 ; 1, and the power applied 50lbs, find the force produced; assuming that the cylinder rod divides the lever in the ratio of 3 : 2. Ans. 2000 lbs.

33. The stem 4 inches long of a common hydrometer is divided into 32 degrees, the graduation being from the top to the bottom; its section is $\frac{1}{20}$th of an inch. There are two liquids whose sp. grs. are 1·3 and 1·7. When immersed in the former, the hydrometer sinks to the point marked 15°, and when in the latter, to 23°. Find the volume of the hydrometer.

Ans. $\frac{49}{160}$ cb. in.

34. In Bramah's press, if A and a be the areas of the cylinders, D and d the distances of the handle and the rod from the fulcrum, P the force applied to the handle, prove that the pressure produced $= P.\dfrac{DA}{ad}$.

35. A hollow cone whose axis is vertical and base downwards is filled with equal volumes of two liquids, whose densities are as 3 : 1. Prove that the pressure at a point on the base is $(3 - \sqrt[3]{4})$ times as great as when the vessel is filled with the lighter fluid.

36. The sides of a solid pyramid are isosceles triangles. The base is a rectangle whose sides are $3a$ and $2\sqrt{5}a$, and the height $2a$. It floats with its base in the surface of a fluid. Prove that the ratio of the whole pressure to the resultant pressure is as $9+5\sqrt{5} : 6\sqrt{5}$.

37. v is the volume of the cylinder of a common pump, and v' that of the water pipe above the surface of water. The water will just rise to the valve at the first stroke, if the height of the water pipe be to that of the water-barometer as $v - v' : v$.

38. In a diving bell, the surface of the water inside is 17 ft. below the surface outside. If h' be the height of the mercurial barometer placed inside, and h its height at the beginning of the descent, prove that $2h' = 3h$, the height of the water barometer being 34 ft.

39. The upper valve of a Smeaton's air-pump opens when the piston is half way up; what was the density of the air in the receiver at first? Show that the distances of the piston from the top of the cylinder when at any stroke the upper valve begins to open, decrease in geometrical progression at every stroke.

40. An ellipse whose focii are S and H, is immersed in a liquid so that its major axis is horizontal, and one extremity B of its minor axis in the surface. Shew that if e be its excentricity, the pressure on the circle inscribed in the triangle SBH is to that on the circle circumscribed as $8e^2(1-e)^3 : 1$.

41. A tight fitting piston of weight w and area k closing a vertical cylinder full of air rests in equilibrium. Shew that if the temperature of the enclosed air be raised by $\dfrac{w}{k\pi a}$ degrees, the piston will ascend to the top of the cylinder, π being the atmospheric pressure, and a the increase of a unit of volume for each additional degree of temperature.

42. A cylindrical diving bell sinks in water till the air occupies $\frac{1}{n}$th of its volume. How much atmospheric air must be forced into it, that it may sink to a further depth of nh ft. without the water rising any higher in the bell, h being the height of the water barometer. Ans. Volume of bell.

43. A piece of teak-wood weighs 320·4 grs. in air. The wood and a sinker together weighs 432·7 grs. in water, and the sinker alone weighs 590 grs. in water. Neglecting the weight of air displaced in the above operations, find the weight of a cubic foot of the teak. Ans. 670·71 oz. nearly.

44. A hollow iron shell containing air at $80^\circ F$ is heated in a furnace to $1000^\circ F$. What is the excess of the pressure inside over that of the atmosphere outside ?

45. The radii of the cylinders of a Bramah's press are 10 and $1\frac{1}{8}$ in. respectively, and a power of $\frac{1}{4}$ ton is applied through a lever whose arms are $1\frac{1}{2}$ in. and 12 in. Find the actual weight that can be raised, deducting 1 per cent. of the total weight for friction. Ans. 157 5 tons nearly.

46. A hollow cone, vertical angle $2\cot^{-1} 2\sqrt{2}$, filled with a fluid, is held with its vertex downwards, and its axis inclined to the horizon at angle θ. Find the normal pressure on the whole surface including the base, and shew that this pressure is maximum when $\cot\theta = \sqrt{2}$

47. Equal masses of 3 liquids whose densities are as 1 : 3 : 5, are disposed in layers in a prism whose base is horizontal and sides vertical Shew that the whole pressures exerted by the 3 fluids on any side of the prism are equal; and if the fluids be mixed together, the total pressure on the side will be increased in the ratio of 23 : 15.

48. A lamina is formed by joining an isosceles triangle with a semicircle whose diameter $2a$ coincides with the base of the triangle. The lamina floats in water in a state of neutral equilibrium, having its axis vertical, and $\frac{4}{5}$ths of the semicircle immersed. If the sp. gr. of the latter be ·4, shew that the sp. gr. of the triangle is $\frac{\pi^2}{20}$, and its altitude $\frac{4a}{\pi}$.

PROBLEMS FROM THE CALCUTTA UNIVERSITY PAPERS 1884-86.

1. If a quadrilateral lamina $ABCD$ having AB parallel to CD, be immersed in liquid with AB in the surface, the center of pressure will be at the intersection of CA and BD, if $AB^2 = 3CD^2$.

2. In the formula $H = C(\log h_1 - \log h_2)$ for finding the difference of level of two stations, in terms of the heights of the barometer at the stations, if $C = 65{,}000$, what height must

we ascend in order that the height of the barometer at the bottom to the height at the top may be as 5 : 4, (log 2 = ·30103).

Ans. 6299·15.

3. A cylindrical diving bell $7\frac{1}{2}$ ft. high is let down in water until its top is 18 ft. below the surface of the water; to what height will the water rise in it, the water barometer standing at 32 ft.? Ans. 3·1 ft. nearly.

4. Given the wright of 1 lb. to be the unit of pressure, one minute the unit of time, and water the standard substance, find the unit of length, assuming $p = g\rho z$. Ans. $4\ (7{\cdot}2)^{\frac{1}{4}}$

5. The sp. gr. of platinum, standard gold and silver are respectively, 21·5, 17·5 and 10 5, and the value of an ounce of each, 35 shillings, 80 shillings, and 4*s.* 3*d*, respectively. Find the value of a coin composed of platinum and silver of the same size and weight as a sovereign. Ans. $7\frac{1}{55}$ sh.

6. A closed hemispherical vessel is just filled with liquid and suspended from a point in the rim of the base; compare the whole pressure on its curved surface with the weight of the liquid. Ans. 57 : $(2\sqrt{73})$.

7. If a triangular prism can float in a liquid with its *C. G.* and one edge simultaneously in the plane of the surface of the liquid, prove that it is isosceles.

8. A portion of a solid sphere is cut off by a plane. Shew that it will float in a liquid of greater density than its own with its plane face immersed.

9. Water is forced to a height of 60 ft. by means of a force pump, of which the piston is 1 ft. in diameter, and has a stroke of 7 ft. Find the horse power required to drive the engine at a speed of 40 revolutions per minute, assuming the piston in its lowest position to be on a level with the free surface of the water. Ans. 25.

10. What degree of exhaustion does a height of 12 inches of mercury in the barometer-gauge indicate, when the barometer stands at 30 inches? Ans. $\frac{2}{5}$.

11. The capacity of the receiver of a condenser is 50 times that of the barrel, and after 40 strokes of the piston, the temperature of the air in the receiver has risen from 30°*C* to 35°*C*. Find the air pressure in the receiver.

Ans. 1·83 nearly.

12. As a diving bell is lowered in a liquid, prove that the tension in the chain is least when the bell is just completely immersed.

13. A spherical bulb of air is released from a bell at a depth x of water, its diameter at that depth being d, and its temperature $t^\circ C$. Find its diameter at a depth y, where its temperature is $T^\circ C$, h being the height of water barometer.

Ans. $\sqrt[3]{\dfrac{(x+h)(1+aT)}{(y+h)\,(1+at)}}\,d.$

14. A sphere floats in a liquid of greater density that its own. Shew that the whole pressure on its surface is the same as that on the curved surface of the circumscribed cylinder immersed to the same depth in the same liquid. The C. G. of the spherical surface cut off by any two parallel planes lies midway between the planes.

15. A hemisphere of density σ floats in a liquid of the same density with its plane surface vertical, and its highest point in the surface of the liquid. Prove that the resultant pressure on its curved surface passes through the center of the sphere, and is equal to $\frac{1}{2}\sqrt{13}$ the times the weight of the hemisphere.

16. A square lamina is immersed in a liquid, with its plane vertical, an angular point in the surface, and a diagonal horizontal. Prove that its center of pressure lies at a depth of $\frac{7}{12}$th of the depth of its lowest angular point.

17. Two equal cylindrical tubes A and B, each 25 inches in length and open at the top, are connected by a horizontal tube fitted with a stop cock. The tubes are filled with mercury to a height of 5 inches in each, the tube A is hermetically sealed at the top, and the stop cock is closed. If the tube B is now filled up to the top with mercury, and the stop cock be afterwards opened, prove that 5 inches of mercury will flow from B to A, the height of the mercurial barometer being 30 inches.

18. A balloon, whose volume is 10,000 cubic feet at a place where the barometer stands at 30 inches and the temperature is 86° F, is released and ascends to an altitude at which the barometer stands at 25 inches and the temperature is 68° F. Find the increase in its volume, taking the coefficient of expansion of the gas for $1^\circ C = \frac{1}{273}$. Ans. 1600 cb. ft. nearly.

19. If P be the resultant pressure on the plane ends of a solid cylinder completety immersed in a fluid, and Q the resultant pressure on the curved surface, shew that the axis of the cylinder is inclined at an angle $\tan^{-1}\dfrac{Q}{P}$ to the vertical.

20. If $\frac{1}{2}$ be the measure of the acceleration of gravity, and 1 that of the density of water, find the units of space and time in order that the formula $W = g\rho V$ may give the weight in pounds.

Ans. $\frac{1}{5}$ ft.; $\frac{1}{8\sqrt{5}}$ sd.

21. Water is poured into a U shaped tube until the legs, which are 15 inches long, are half full; as much oil is then poured into one of the legs. What length of the tube does it occupy, the sp. gr. of the oil being $\cdot 75$? Ans. 12 ft.

22. If H be the position of the center of pressure of a plane area immersed vertically in a fluid, when G, the center of gravity is at a depth h, and if H' be its position when the depth has increased to h', prove that $GH' = GH\frac{h}{h'}$.

23. If a triangle can float in a liquid with its plane vertical, and one side horizontal, shew that it must be isosceles.

24. A cylindical diving bell is lowered into water until $\frac{3}{4}$ths of it is filled with water, shew that if d be the depth of the top of the bell below the surface, the height of the bell is $4(3h - d)$, where h is the height of the water barometer.

25. If σ be the sp. gr. of a substance found by the use of the hydrostatical balance, when the sp. gr. of the air is neglected, and S be the sp. gr. of air relatively to water, shew that the error in σ will be $S(\sigma - 1)$.

26. If a solid hemisphere is moveable about the center of its plane base which is fixed in the surface of a liquid, and if the density of the liquid be twice that of the solid, prove that any position of the solid will be one of equilibrium.

27. Since a homogeneous hemisphere can, having its plane face vertical, float in a fluid of twice its sp. gr.; shew that if D be the depth of the center of pressure of the immersed semicircle, and d the distance of the center of mass of the hemisphere from the center of its plane base, and d' the distance of the center of mass of a semicircular plate whose radius is r from its center, then $\frac{3}{8}d.r = D.d'$.

28. In the formula for the measurement of heights by a barometer, calculate the value of the coefficient $\frac{k}{g}$, assuming that at 0°C, and under 30 inches of mercury pressure, 100 cub. inches of dry air contains 29·3456 grains, whereas 1 cub. inch of water contains 252·5 grains, and the sp. gr. of mercury is 13·596.

29. A square plate floats in water with its plane vertical and one angular point below the surface, find its position of equilibrium.

30. A triangular plate is immersed in a vertical position, its base horizontal, and its vertex in the surface of the water; draw 7 lines parallel to the base so that the pressures on the 8 areas may be equal.

Ans. The lines are at depths $\frac{1}{2}$, $\frac{1}{2}\sqrt[3]{2}$, $\frac{1}{2}\sqrt[3]{3}$ &c. of the vertical height of the triangle.

31. If the atmospheric pressure at the surface of the earth be 14 lbs. on the square inch, and the height of a column of air of the same uniform density as at the surface, which would produce the same pressure, be 5 inches; shew that the density of the air is $\frac{1}{800}$ nearly, the density of water being 1.

32. A rectangle is immersed in a liquid with one side in the surface and its plane vertical, shew that the depth of the center of pressure will be $\frac{2}{3}a - \frac{ab}{3(a+2b)}$, where b is the height of the water barometer, and a the length of a vertical side of the rectangle.

33. An uniform beam AB, capable of turning about a fixed point at the extremity A, rests with a portion BM immersed in water; if s denote the sp. gr. of the beam, prove that $\sqrt{(1-s)} = \frac{AM}{AB}$, and that the pressure on the fixed point $= \frac{AM}{AB}$, × weight of the fluid displaced.

34. The readings of a faulty barometer containing some air are 29·4 and 29·9 inches, the corresponding readings of a correct instrument being 29·8 and 30·4 inches respectively; prove that the length of the tube occupied by the air is 2·9 inches, when the reading of the faulty barometer is 29 inches; and find the corresponding correct reading.

Ans. 29·29 nearly.

35. In one Smeaton's air pump, the volume of the barrel is $\frac{1}{10}$ of that of the receiver, and in another it is $\frac{1}{5}$ of it; shew that the densities of the air in the two receivers after 3 ascents of the piston are as $12^3 : 11^3$.

APPENDIX I.

Formulæ to be remembered in the working of examples.

C. G. of a Semicircle is on its axis at the distance $\frac{2r}{3\pi}$ from the centre.

,, of the surface of a cone is on its axis at distance of $\frac{2}{3}h$ from the vertex.

,, of a parabola cut off by a double ordinate, is on its axis at distance of $\frac{3}{5}h$ from the vertex.

,, of a solid hemisphere $= \frac{3}{8}r$ from center,

,, of a solid paraboloid is on the axis at distance of $\frac{2}{3}h$ from the vertex.

,, of a solid cone is on its axis at distance of $\frac{1}{4}$th of its length from base.

,, of the surface of a hemisphere is $\frac{1}{2}r$ from center.

,, of a spherical surface cut off by any two parallel planes, lies midway between the planes.

Volume of a sphere $= \frac{4}{3}\pi r^3$.

,, of cone or pyramid $= \frac{1}{3}$ the base $\times$ height.

Surface of sphere $= 4\pi r^2$.

,, of a cone $= \pi r^2 \operatorname{cosec} a = \pi r\sqrt{(r^2 + h^2)}$, r being radius the base, h its altitude and a its semi-vertical angle.

Area of an ellipse $= \pi a.b$, a and b being semi-axes.

,, of a parabola cut off by an ordinate $= \frac{2}{3}$ ordinate $\times$ abscissa.

APPENDIX II.

A SHORT HISTORY OF HYDROSTATICS.

The true doctrine of Hydrostatics rests on the distinct idea of fluid as a body of which the parts are perfectly moveable among each other by the slightest pressure, and in which a pressure exerted at one point is transmitted without change to all other points. From this idea of fluidity necessarily follows that multiplication of pressure, which for a long time puzzled the minds of ancient philosophers, and even to this day goes by the name of *the hydrostatical paradox.* Indeed the conception and retention of the idea of perfct fluidity so as to follow it in all its consequences, was by no means an easy process. The non-apprehension of this idea and its consequences led even such great men as Aristotle to many erroneous speculations about heavy and light things. The true idea seems to have first occured to the celebrated Archemedes of Syracuse, who in the third century before Christ, laid the foundation of the science of hydrostatics in his *Treatise on floating bodies.* In this remarkable work, besides enunciating the laws of fluid action and the condions of equlibrium of floating bodies, he solves problems of considerable complexity. Tradition has preserved a story characteristic of the inventive genius of a great mind like that of Archimedes. It is said that Hiero, King of Syracuse, having suspected unfair play in a crown of gold which he had caused to be made, commissioned Archimedes to solve the difficulty. The philosopher long thought over the problem, when one day while taking his bath, he noticed that his body was displacing a volume of water exactly equal to its own. By a train of reasoning pecular to himself, it struck him that equal weights of different substances must displace different volumes of water, and gold being specifically heavier than any known alloy, a crown of pure gold ought to displace less quantity of water than an impure one of the same weight. It is said that he was so much elated with this discovery, that unable to contain himself, he immediately ran almost naked into the street, crying out *Eureka Eureka*, I have solved it, I have solved it !!

But the general mind was not yet ripe for grasping the truths which Archimedes had realized. After being clearly awakened in the mind of Archimedes, the ideas were destined to sleep for many centuries, till they were again called up in Galileo, and more remarkably in Stevinus. The intermediate time